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2056. (The Editor.)—Prove that the straight lines joining the in- and circum-centres will subtend a right angle at the orthocentre if

$$1+(1-2\cos A)(1-2\cos B)(1-\cos C) = 8\cos A\cos B\cos C \ldots. 78$$

2061. (The Editor.)—Prove that the envelope of the radical axis of two circles of curvature of the ellipse $a^2y^2+b^2x^2=a^2b^2$, drawn at the ends of conjugate diameters, is the sextic (of class 6)

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left\{2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-3\right\}=\left\{3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-4\right\}^2,$$ having

asymptotes $$2\left(\frac{x}{a}\pm\frac{y}{b}\right)\pm 3=0;$$

four cusps, $x^2=2a^2$, $y=0$; $y^2=2b^2$, $x=0$; four acnodes, $\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{2}{3}$; and two crunodes at infinity. 91

2140. (The Editor.) — Trace the curve whose equation is $x=2a\sin\frac{y}{x}$; and prove that each loop has the same area πa^2, and is bisected by the straight line that joins the origin to the point where the tangent is parallel to the axis of y. 81

2672. (R. Tucker, M.A.)—From a point on an ellipse chords are drawn parallel to fixed straight lines; find the maximum triangle formed by the three chords of section. 41

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5106. (J. F. Moulton, M.A.)—A ray of light is refracted through a prism in a principal plane. Show that, if the dispersion of two neighbouring colours be a minimum,

$$\frac{\sin(3\phi' - 2i)}{\sin\phi'} = 1 - \frac{2}{\mu^2}. \quad \ldots\ldots\ldots\ldots \quad 34$$

5132. (Professor Crofton, F.R.S.) — From any point P within a triangle lines are drawn parallel to the two sides, meeting the base on D, E. Show (1) [as may be done without integration] that the mean value of the triangle PDE is one-sixth of the former triangle; and find (2) the mean value of the segment DE..... 27

5235. (Rev. W. A. Whitworth, M.A.)—Prove that the total number of signals which can be made with n different flags on s different masts is one less than the coefficient of x^n in the expansion of

$$n!\,\epsilon\,(1-x)^{-s}. \quad \ldots\ldots\ldots\ldots \quad 33$$

5236. (R. A. Roberts, M.A.) — Prove that the circumcircle of a triangle inscribed in a conic and circumscribed to a circle touches two fixed circles having double contact with the conic..... 46

5300. (R. Tucker, M.A.)—Particles are projected with a constant velocity from the same point; through the focus a straight line is drawn parallel to the direction of projection, cutting the path in P, Q. Prove that the locus of P, Q is a sphere with radius equal to twice focal distance of point of projection, and tangential to the directrix plane. 90

5301. (Rev. E. Hill, M.A.) — Certain persons have imagined the existence of a subterranean connexion between the waters of the Dead Sea and the Mediterranean. Although the difference of their levels is 1300 feet, yet, since the ratio of their densities is 1·24, it is possible that such a passage may exist. But find its necessary depth..... 85

5322. (J. L. McKenzie, B.A.) — A triangle ABC is inscribed in a conic; through C a line is drawn parallel to the tangent at A, and meeting the conic in D; through A a line is drawn parallel to BC, and meeting the conic in E: prove that DE is parallel to AB. 29

5323. (Rev. E. Hill, M.A.)—The radius of a thermometer bulb is $\cdot^{0}$ of an inch, that of the stem ·01, height of the freezing point above the bulb 1 inch, and of the boiling point $7\frac{1}{2}$ inches; show that the expansion of mercury for a degree centigrade is about $\frac{1}{5550}$ of its volume. 111

5346. (W. E. Wright, B.A.)—If $\dfrac{\sin\alpha}{\sin(\alpha - 2\beta)} = c$, find $\int \sin\alpha\, d\beta$. 110

5411. (A. Martin, LL.D.)—Find the probability that a random shot will hit a target a feet square at a distance of b feet. 111

5485. (R. A. Roberts, M.A.) — If tangents are drawn to a cubic from any point of a harmonic polar, prove that they form a pencil in involution. 60

5646. (R. Tucker, M.A.)—Normals to a lemniscate are drawn at points whose vectorial angles are θ and $\frac{3}{2}\pi + \theta$; find the locus of their intersections and the area of the curve..... 110

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5652. (J. Hammond, M.A.)—If

$$\phi(x) = A_0 + A_1 x + A_2 x^2 + \ldots \text{ ad inf.},$$

sum the infinite series

$$A_0 + \tfrac{1}{2}(A_0 + A_1) + \tfrac{1}{3}(A_0 + A_1 + A_2) + \tfrac{1}{4}(A_0 + A_1 + A_2 + A_3) + \&c.,$$
$$A_0 - \tfrac{1}{2}(A_0 - A_1) + \tfrac{1}{3}(A_0 - A_1 + A_2) - \tfrac{1}{4}(A_0 - A_1 + A_2 - A_3) + \&c.$$

Supposing the series for $\phi(x)$ to be convergent, find the condition that the first of these derived series shall be convergent. 39

5776. (R. A. Roberts, M.A.)—Find the locus of the centroid of an equilateral triangle (1) inscribed in a conic, (2) circumscribed about a conic, (3) self-conjugate with regard to a conic. 33

5785. (D. Edwardes.)—A train goes round a curve, the radius of the curve lying midway between the metals being r. If m be the mass of the train, v its velocity, and a the breadth of the gauge, find the pressure on the metals, the outer metal being raised so that the pressure is perpendicular to the floor of the carriage. 102

5966. (Elizabeth Blackwood.)—Find the average area of the circle that passes through three points taken at random on the surface of a sphere. 53

6014. (Professor Matz, M.A.)—There is a circular field whose diameter is $2m$ ($= 100$) feet; find the length of a rope fastened at a point n ($= 60$) feet from the centre of the field, in order that a cow, tied by the tail to the rope, can graze over exactly one-a^{th} ($= \frac{1}{10}$) of an acre of the field. 31

6016. (The Editor.)—Find the probability that a circle drawn at random on the surface of a sphere will cut off a portion of spheric surface less than one-fourth of the surface of the sphere. 28

6490. (Professor Hudson, M.A.)—A series of lemniscates are drawn having the same axis and node, and another series with the same node and different axis; prove that any lemniscate of the first set cuts any of the second at an angle which is twice the angle between the axes of the two sets. 74

7306. (Professor Hudson, M.A.)—From a point P on a parabola, focus S, PM, PN are drawn perpendicular to the directrix and axis; PT, PG are the tangent and normal limited by the axis. What line represents the resultant of forces represented by PM, PT, PS, PN, PG?... 59

7381. (A. Martin, M.A., LL.D.)—If $F(x)$ denote any function of x, prove that, when n is greater than any assignable quantity,

$$\int [F(x)]\,dx = x/n\,[F(x/n) + F(2x/n) + F(3x/n) + \ldots + F(nx/n)].$$

..... 38

7520. (The Editor.)—Through a given point R, draw, geometrically, a straight line RDEF, cutting a straight line in D, and a given circle in E, F, so that RD may be equal to EF. 65

7561. (W. Perryman.)—If $xy = 1$ be the equation to the rectangular hyperbola, prove that the difference between the length of the infinite arc of the curve from its vertex, and that of its asymptote from the

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centre, is equal to $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{14}{15}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot\frac{20}{21}$..., and its value to the fifteenth decimal place is ·847213084793979; or, if the curve be severed at vertex and be made to coincide with the asymptote by becoming straight, the point of severance will fall short of the centre of the curve by the above value. 96

7674. (Professor Hudson, M.A.)—From the top of a hill the depression of a point on the plain below is 30°, and from a spot three-quarters of the way down the depression of the same point is 15°; find within 1′ the inclination of the hill. 53

7721. (Belle Easton.)—One card out of a pack has been lost. From the remainder of the pack, thirteen cards are drawn at random, and are found to consist of two spades, three clubs, four hearts, and four diamonds. What are the respective chances that the missing card is a spade, a club, a heart, or a diamond? 44

7858. (R. Knowles, B.A.)—From a point Q in the circumference of an ellipse, chords PQ, PR are drawn at right angles; (1) show that the locus of the pole of the chord QR is a right line; (2) as P moves along the ellipse, find the locus of the intersection of this line with the tangent at P; (3) show that this locus is also that of the pole of normal chords to the ellipse. 45

8180. (W. J. Greenstreet, B.A.)—If A_p is coefficient of x^p in expansion of $(a+bx+cx^2)^n$, prove that, if $m>n$, $c^m A_{n-m} = a^m A_{n+m}$.... 44

8488. (Professor Orchard, B.Sc., M.A.)—Find the current given by a battery of 15 equal and similar cells in 5 rows of 5, 4, 3, 2, 1 cells respectively: the electro-motive force of each cell being 1·6416, and the resistance of the wire joining the poles being the same as that of one cell. 71

8947. (The late W. J. C. Sharp, M.A.)—Prove that the lines of curvature on a developable surface are either tangents or evolutes of the cuspidal edge..... 35

8973. (The late W. J. C. Sharp, M.A.)—If

$$(n)^{(r)} \text{ denote } n(n-1)\ldots\ldots(n-r+1),$$

show that $n^r = (n)^{(r)} + {}^rA_1 (n)^{(r-1)} + {}^rA_2 (n)^{(r-2)} + \&c.,$

where
$${}^rA_p = \frac{(r-p)^{r-1}}{(r-p-1)!} - \frac{(r-p-1)^{r-1}}{(r-p-2)!} + \frac{(r-p-2)^{r-1}}{2!\,(r-p-3)!} - \frac{(r-p-3)^{r-1}}{3!\,(r-p-4)!} + \&c.$$
$$= \Delta^{r-p} 0^r \div (r-p)!$$
..... 36

9110. (Professor Asutosh Mukopadhyay, M.A., F.R.A.S.)—Prove that (1) the locus of the points on a system of confocal ellipses or hyperbolas, for which the eccentric angle has a constant value, is a hyperbola; (2) the envelopes of the hyperbolas, for different values of the eccentric angle, are two pairs of straight lines, passing through the foci and parallel to the bisectors of the angles between the axes, and are real when the confocals are hyperbolas, but imaginary when they are ellipses. 48

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9134. (Prof. Mahendra Nath Ray, M.A., LL.B.) — Show that the radius of gyration of a circular arc AB of uniform density about a line through its middle point at right angles to its plane, is represented by the chord of half the arc cut off by the line drawn through the centre of gravity of the arc parallel to the chord AB..... 36

9296. (A. Kahn, B.A.) — Find the developable whose edge of regression is the curve of intersection of the two surfaces

$$x^2-y^2-4z^2=0, \quad yz+zx-a(x-y)=0 \text{.....(1, 2).}$$

..... 80

9404. (Rev. T. C. Simmons, M.A.)—If the tangent at any point P of a parabola meet the tangent at the vertex A in T, and the diameter TQV meet the curve in Q and AP in V; then, drawing PM at right angles to AT, and the tangent at Q to meet PM in R and AM in W, the ratios of (1) PV to PA, (2) TQ to TV, (3) triangle RMW to triangle PMA, (4) curvilinear area AQPV to triangle ATP, are all constant. Prove that these properties hold for all parabolic curves ($y = cx^m$), and that the respective ratios are denoted by (1) m, (2) $(1-m^{-1})^{m-1}$, (3) $(1-m^{-1})^{m-1}$, (4) $m/(m^2-1)$..... 84

9545. (Professor Kalipada Basû, M.A.) — Prove that, when n is indefinitely increased, the area of the loop of the curve

$$a^{2n-2}y^2 = n^3x^{2n}\frac{a-x}{a+x} \quad \text{is} \quad (\tfrac{1}{3}\pi)\,a^2. \text{.....} \quad 109$$

9983. (Professor Ignacio Beyens.) — Si (S) désigne la somme des surfaces des cercles de Neuberg correspondantes à chacun des côtés d'un triangle ABC, (e) l'excentricité de l'ellipse de Brocard, on aura

$$S = \pi(a^2+b^2+c^2)\,e^2/(1-e^2). \text{.....} \quad 95$$

10191. (Professor Tharaud.)—Étant données deux droites rectangulaires OX, OY, on prend sur OY les points A, B, et sur OX le point C, tels que $OA = mOB = mOC = a$, m étant un nombre positif quelconque et a une longueur donnée. (1) Trouver sur la droite indéfinie BC un point M tel que le rapport de ses distances au point A et à la droite OX ait une valeur donnée λ; (2) discuter en laissant m constant et faisant varier λ: étudier comment sont placés les points trouvés par rapport aux points B et C; (3) il y a en général deux points M et M′ satisfaisant au problème: déterminer λ de manière que ces points se confondent, et chercher entre les coordonnées du point Q correspondant à ce cas, une relation indépendante de m; (4) en partant de la relation précédente, construire le lieu décrit par le point Q lorsque la droite BC se déplace parallèlement à elle-même. 58

10208. (J. J. Barniville, M.A.)—Inscribe in a quadrant a square having two angles on the circumference. 82

10583. (J. MacNeill, M.A.) — Find the contents, in gallons, of a circular-section basin, 6 feet in perpendicular height, the diameters being:—at base, 1 foot; at 1 foot perpendicular height, 2 feet; at 2 feet perpendicular height, 4 feet; and so on, the diameter being doubled for every foot of perpendicular height. 111

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10675. (R. Knowles, B.A.)—Prove that the four points of intersection of the opposite sides of an octagon, inscribed in a conic, are collinear. 94, 106

10746. (R.) Knowles, B.A.) — A hexagon is inscribed in a conic; prove that the points of intersection of the diagonals of the two quadrilaterals formed by the sides of the hexagon and one of its diagonals, and the point of intersection of the diagonals of the corresponding circumscribed hexagon, are collinear. 103

10958. (Professor de Longchamps.)—Soient AB, BC, CA les directions positives des côtés d'un triangle ABC. Par les sommets, on mène des droites AP, BQ, CR, faisant, avec ces directions positives, un même angle α. Les droites considérées forment un triangle A'B'C'; démontrer que (1) le centre du cercle circonscrit à A'B'C' coincide avec l'orthocentre de ABC; (2) tout point remarquable de A'B'C' décrit une circonférence. Toutes ces circonférences passent par un même point, l'orthocentre de ABC. Elles constituent un *réseau* de circonférences remarquables relativement au triangle ABC. 66

11089. (Professor Ramaswami Aiyar.)—Given a conic referred to its axes, show that constants λ, μ may be determined, such that the circle passing through any point (h, k) on the conic, and having its centre at $(\lambda h, \mu k)$, may cut the conic again at the vertices of an equilateral triangle. 41

11108. (R. Tucker, M.A.) — (1) OPQ is a triangle formed by the intersections of a transversal PQ with the asymptotes OP, OQ of a hyperbola. If $\left.\begin{matrix} OP = h, \ OQ = k, \\ \angle POQ = 2\alpha, \end{matrix}\right\}$ prove that, if the "S." point of the triangle be on the curve, then $h^3k^3 = \lambda^2[h^2 + k^2 - hk\cos 2\alpha]^2$, where $\lambda^2 = a^2 + b^2$. Find also the locus of the centroid and circumcentre. (2) OPQ is a triangle formed by the tangents OP, OQ to a parabola and the chord of contact PQ; prove that, if the "S." point of the triangle is on the curve, then $PQ^2 = 2\,OP \cdot OQ$..... 105

11109. (A. Gordon.)—If

$$Q_{r+2} - (a+b)Q_{r+1} + abQ_r = 0, \quad ab. \ P_{r+2} - (a+b)\,P_{r+1} + P_r = 0,$$

and

$$P_0 = Q_0 = 0, \quad P_1 = Q_1 = 1;$$

prove that $P_1Q_1 - 3P_3Q_3 + 5P_5Q_5 \ldots (-1)^m\,(2m-1)\,P_{2m-1}\,Q_{2m-1}$ is

$$\frac{ab\,(-1)^m}{(a-b)^2}\left\{2m - 2m\,\frac{a^{4m} + b^{4m}}{(a^2+b^2)(ab)^{2m-1}} + \frac{(a^2-b^2)(a^{4m}-b^{4m})}{(a^2+b^2)^2\,(ab)^{2m-1}}\right\}. \ \ldots \ 63$$

11121. (G. Heppel, M.A.)—In several books on Geometrical Drawing the following construction is given for inscribing a regular polygon of n sides in a circle:—Divide a diameter into n equal parts. From the vertex of an equilateral triangle on the diameter draw a line passing through the *second* point of division. The intersection of this with the circle determines a vertex of the polygon. Show what is the reason for this construction, and examine how far it is accurate for different values of n. 93

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11162. (Professor Zerr, M.A.)—Prove that the radius of curvature at any point on the ellipse or hyperbola is equal to the product of the focal distance divided by the perpendicular from the centre on the tangent to the point. 35

11165. (The late Professor Clifford, F.R.S.)—A rigid body is made by fixing the flat side of a hemisphere to one of the flat ends of a portion of a circular cylinder of equal radius cut out by planes perpendicular to the axis, the whole being of uniform density; find (1) the time of small oscillations about a diameter of the other flat, circular end; and, if the body rotates about an axis through the centre of the hemisphere, making an angle θ with the axis of the cylinder, determine (2) the moment of momentum..... 26

11173. (Professor Laisant.) — Soient CA, CB deux demi-diamètres conjugués d'une ellipse, et F l'un des foyers; on construit le parallélogramme FAVB. Démontrer que (1) la droite CV est dirigée suivant la bissectrice de l'angle AVB; (2) $CV^2 = 2VA \cdot VB$; (3) le lieu du point V est une ellipse quand on fait varier le système des diamètres conjugués. Etendre ces propriétés à l'hyperbole. 103

11181. (Professor Bénézech.)—Trouver quatre nombres entiers tels que leur somme soit égale à la somme obtenue en ajoutant le produit du plus grand par le plus petit nombre au produit des deux autres. 41

11185. (R. W. D. Christie.)—Prove that

$$a^r + (a+d)^r \ldots\ldots \{a+(p-1)d\}^r = p(\mathrm{M}),$$

where $r = 2$ or 5, and $p =$ any prime > 3.

[*Ex. gr.*, $3^2 + 7^2 + 11^2 + 15^2 + 19^2 + 23^2 + 27^2 = 7(\mathrm{M})$,

also $2^5 + 4^5 + 6^5 + 8^5 + 10^5 + 12^5 + 14^5 = 7(\mathrm{M})$.] 109

11218. (Professor Genese, M.A.) — The bisectors of the angles between opposite sides of a quadrilateral inscribed in a circle meet on the line joining the middle points of the diagonals, and divide that line internally and externally in the ratio of the diagonals. 73

11240. (R. Knowles, B.A.)—ABCDE is a pentagon inscribed in a conic; the five points of intersection of the sides AB, BC, CD, DE, EA with the tangents at D, E, A, B, C respectively are collinear; and of course its reciprocal is true, that the five lines joining the poles of AB, BC, &c. to the points D, E, &c., are concurrent. 43

11242. (W. J. Greenstreet, M.A.)—Draw three circles, each passing through two angular points of a triangle and cutting its incircle orthogonally. Take the triangle formed by the joins of the centres of these circles, and prove that (1) its sides are parallel to contact joins of incircle; (2) the centres of the circumcircles of both triangles are identical; and (3) the radii of the circumcircles differ by $\frac{1}{2}$ radius of the incircle. 87

11243. (Morgan Brierley.)—Given the vertical angle, the perimeter, and the length of a line drawn from the vertical angle to meet the base and divide it in a given ratio, show how to determine the triangle by the properties of plane geometry without reference to conics..... 89

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11261. (Professor Keelhoff.)—Etant donnée une circonférence O, on prolonge un rayon quelconque OM de la longueur $MN = n \cdot MP$, où MP est la perpendiculaire abaissée de l'extrémité M du rayon sur un diamètre fixe AA'. (1) Démontrer que le lieu du point N est un limaçon de PASCAL; (2) trouver l'aire de la courbe; et (3) trouver la longueur de la courbe, dans le cas de $n = 1$. 31

11266. (Professor Cavallin.)—A circle through the centre of a rectangular hyperbola cuts the curve in the points A, B, C, D. Prove that the circle circumscribing the triangle formed by the tangents at A, B, C passes through the centre of the hyperbola, and has its centre at the point on the hyperbola diametrically opposite to D..... 34

11273. (A. E. Daniels, B.A.)—Prove that the area of the triangle formed by the three normals to an ellipse at the points whose eccentric angles are α, β, γ, is given by

$$\frac{(a^2-b^2)^2}{4ab} \tan \tfrac{1}{2}(\beta-\gamma) \tan \tfrac{1}{2}(\gamma-\alpha) \tan \tfrac{1}{2}(\alpha-\beta) \times \left[\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\right]^2.$$

..... 39

11276. (D. Edwardes, B.A.)—Given that

$$x/(m^2\alpha^2-\beta\gamma)^{\frac{1}{2}} = y/(m^2\beta^2-\gamma\alpha)^{\frac{1}{2}} = z/(m^2\gamma^2-\alpha\beta)^{\frac{1}{2}},$$
$$m^2(\alpha^3+\beta^3+\gamma^3) = (1+2m^3)\,\alpha\beta\gamma,$$

prove that $\alpha/(m^2x^2-yz)^{\frac{1}{2}} = \beta/(m^2y^2-zx)^{\frac{1}{2}} = \gamma/(m^2z^2-xy)^{\frac{1}{2}}$,

$$m^2(x^3+y^3+z^3) = (1+2m^3)\,xyz. \text{ ..... } 77$$

11278. (D. Biddle.)—The bases of three right cones, resting on a horizontal plane, touch each other, and are 300, 200, 100 square feet in area, whilst the respective heights are 100, 400, 300 feet (linear). A marble falls from a point above the centre of gravity of the three cones and one-third higher: find where it strikes the plane. 61

11279. (Rev. T. Roach, M.A.)—Prove that, in an equation of the n^{th} degree, expressed in its simplest form, if p be the numerical value of the greatest negative coefficient, and x^{n-r} the highest power of x which has a negative coefficient, then $(1+p)^{1/r}$ is a superior limit of the positive roots. 37

11287. (R. Knowles, B.A.)—Prove that the sum of the series

$$\frac{1}{4} - \frac{2^3}{7} + \frac{2^6}{10} - \frac{2^9}{13} + \dots \textit{ ad infin.} \equiv \frac{1}{16}\left(2 - \tfrac{1}{3}\log 3^{\frac{1}{2}} - \tfrac{1}{6}\pi \cdot 3^{\frac{1}{2}}\right).$$

..... 46

11294. (Professor Sollertinsky.)—Si le sommet C d'un triangle ABC décrit une circonférence Δ et la base AB soit fixe, l'aire ABC varie proportionnellement à la puissance de C par rapport à un cercle fixe Δ'. Dans quel cas ce cercle Δ' peut-il être remplacé par un point fixe?... 51

11312. (J. C. St. Clair.)—Show that, of the three systems in involution determined by four collinear points taken in pairs, the centres of any two are conjugate points of the third. 47

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11322. (R. Knowles, B.A.)—From a point T on the latus-rectum of a parabola, tangents are drawn to meet the parabola in P, Q; the circle TPQ cuts the curve again in CD; the diameter of the parabola through T meets the curve in R; K is the pole of CD, with respect to the circle; O is the centre of curvature at R; prove that OK is a diameter of the parabola..... 42

11347. (R. Lachlan, M.A.)—Show that the number of ways in which n boys may be arranged, in (1) a row, (2) a circle, so that n particular boys may be separated, is

$$\frac{(n-r)!\,(n-r+1)!}{(n-2r+1!}, \quad \frac{(n-r)!\,(n-r-1)!}{(n-2r)!}. \quad \ldots\ldots\ldots\ldots 38$$

11348. (Morgan Brierley.)—Find x, y, z, from

$$(x+y)z = a\,(=56, \text{ say}), \quad (x+z)y = b\,(=50), \quad (x+y)x = c\,(=36).$$

..... 38

11350. (Belle Easton, B.Sc.)—AB is the vertical diameter of a fine circular tube in which move three equal particles P, Q, Q′ (modulus of restitution = 1 for any pair); P starts from A, and Q, Q′ in opposite senses from B, with such velocities that at the first impact all three have equal velocities; prove that throughout the motion the line joining any pair is either horizontal or passes through one of two fixed points, and that the intervals of time between successive impacts are all equal. 55

11353. (H. W. Curjel, B.A.)—A soap bubble B, of radius b, is inside another bubble A, of radius a. The bubbles are acted on by no external forces except the pressure of the air. Find the effect of the bursting (1) of A, (2) of B, on the size of the other, the surface tension of the soap solution and the atmospheric pressure being known, and the thickness of the film being neglected..... 128

11356. (E. M. Langley, M.A.)—ABC is an equilateral triangle, D a point in BC such that BD is one-third of BC, and E is a point in AB equidistant from A and D. Show geometrically that CE = EB + BD. 85

11357. (G. E. Crawford, B.A.)—If $\frac{az+b}{cz+d} = u \equiv \xi + i\eta$ and $z \equiv x + iy$, prove that, if (x, y) describes a circle, so does (ξ, η), the four constants being all real..... 62

11375. (Professor Zerr.)—Find (1) the maximum rectangle, (2) the maximum parallelopiped, inscribed in, first, the curve $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$, second, the surface $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} + (z/c)^{\frac{1}{2}} = 1$, $a > b > c$. 108

11376. (Professor Curtis, M.A.)—Any two chords of an equiangular hyperbola cutting each other perpendicularly have the rectangles under their sections equal and opposite in sign. Prove this, and derive the following:—An equiangular hyperbola and a circle, osculating, cut off equal and opposite portions from their common normal. [This latter proposition, we believe, is to be found in the German translation of SALMON's *Conics*. The above is a geometrical proof of it.] 67

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11377. (Professor Neuberg.)—Soit A'B'C' le triangle formé par les tangentes aux points A, B, C de l'hyperbole de KIEPERT du triangle ABC. Démontrer (1) que A', B', C' sont les intersections des droites joignant les milieux des côtés de ABC, avec les droites joignant les pieds des hauteurs de ABC; (2) que les droites AA', BB, CC' sont perpendiculaires à la droite joignant le centre de gravité de ABC à l'orthocentre; (3) que l'orthocentre du triangle A'B'C' coincide avec le centre du cercle des neuf points de ABC. 56

11381. (Professor Michel.)—Soit AC une corde mobile dans une circonférence Δ, de rayon R. Sur AB on prend un point I tel que

$$IA^2 + IB^2 + K \cdot AB^2 = 2(1+2K)R^2;$$

K désignant une constante donnée. Trouver le lieu décrit par ce point I. 36

11382. (Professor Tissot.)—Deux circonférences étant donnés ainsi qu'une droite qui ne les rencontre pas, on peut mener, d'un point quelconque de la droite, une tangente à chaque circonférence. Trouver le point pour lequel la somme des deux tangentes ainsi obtenues est aussi petite que possible, et celui pour lequel leur différence est aussi grande que possible. 72

11383. (Professor Steggall.)—There are twelve electors and three candidates A, B, C for an office. On being put up together, A gets more votes than B, B than C. On being put up in pairs, A is preferred to B, B to C, and C to A; what were the original votes? 75

11389. (D. Biddle.)—Around the wheel of an Attwood's Machine, for illustrating the laws of gravitation, passes a rope, to one extremity of which is attached a weight A, and to the other extremity a pulley B, having weights a and b playing around it. Supposing

$$b = a(g+1)/g-1), \quad \text{and} \quad B + a + b = A(g+1)/(g-1),$$

find the positions of b and B respectively at the end of each successive second, friction and the resistance of the air being disregarded. 49

11393. (H. W. Curjel, B.A.)—If p is a prime not less than 5, show that the number of primes greater than p and less than p^2 is not less than p. 127

11394. (H. W. Segar, B.A.)—There is a series of integers such that, if they be denoted by u_1, u_2, u_3 ..., and their successive differences by v_1, v_2, v_3, ..., then the following relations hold:—

$$u_{2n} = u_n^2 + v_n^2, \quad u_{2n+1} = u_{n+1}^2 + v_n^2, \quad v_{2n} = u_{n+1}^2 - u_n^2, \quad v_{2n+1} = v_{n+1}^2 - v_n^2.$$

..... 79

11395. (J. L. Mackenzie, B.A., B.Sc.)—Prove that $(x+y)^n - x^n - y^n$ is divisible by x^2+xy+y^2, when n is any odd number not a multiple of 3. 112

11405. (Professor Crofton, F.R.S.)—Prove that

$$\frac{x + 2^r \frac{x^2}{1.2} + 3^r \frac{x^3}{1.2.3} + \&c.}{1 + x + \frac{x^2}{1.2} + \&c.} = x + \frac{\Delta^2 0^r}{1.2}x^2 + \frac{\Delta^3 0^r}{1.2.3}x^3 + \ldots x^r \ldots \quad 45$$

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11406. (Professor Zerr.)—Three points are tak n at random in a sphere, and joined to the extremity of a fixed diameter; find the average volume of the tetrahedron thus formed. 107

11414. (Professor Mannheim.)—Sur la médiane am d'un triangle abc, comme diamètre, on décrit une circonférence de cercle. Cette courbe coupe, au point g, le cercle circonscrit au triangle abc; démontrer que les droites ab, ac, ag, et la hauteur ah du triangle, forment un faisceau harmonique. 81

11417. (Professor Gelin.)—Dans tout triangle ABC on a

$$(a\tan B-2\gamma)(b\tan C-2\alpha)(c\tan A-2\beta)$$
$$=-(a\tan C-2\beta)(b\tan A-2\gamma)(c\tan B-2\alpha),$$

α, β, γ étant les rayons des cercles ex-inscrits respectivement dans les angles A, B, C. 128

11424. (G. E. Crawford, M.A.)—If ABC be a triangle, and AA′, BB′, CC′ be three diameters of the circumscribing circle, prove that the sum of the projections of BC, CA, AB on AA′, BB′, CC′, respectively, vanishes, if the proper convention of signs be adopted. 59

11427. (R. Lachlan, M.A.)—If the points of contact of the three tangents which can be drawn from the point P to the cardioid $r=a(1+\cos\theta)$ be collinear, prove that (1) the locus of P is the circle $r+2a\cos\theta=0$; and (2) if the feet of the three normals which can be drawn from P be collinear, the locus of P is the circle $3r=a\cos\theta$. 42

11432. (I. Arnold.)—Construct an equilateral triangle that shall be equal to a given parallelogram. 87

11435. (G. J. Griffiths, M.A.)—Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime is $15\pi^{-2}$. 79

11442. (Professor Shields, M.A.)—A lady owned and had her residence R in the centre of a round island of land, which had a circular field FF around its edge, thus enclosing a round piece of woodland W. One-half of the circumference of the round piece of woodland measures $570\frac{1}{2}$ rods more than its diameter, and the difference between the circumference of the whole island and its diameter is $3769\frac{27}{125}$ rods. She gave her daughter four circular fields D that just touch each other, and thus enclose one acre of land for her residence, which she also gave her; and gave her son a square piece of land S, containing the largest round field possible, the land lines circumscribed about the circular field in such a manner as to enclose one acre of land in each corner outside the circular field. Give the area of the circular field FF, and give dimensions of each child's tract. 93

11448. (Professor Catalan.)—On prend p nombres, dans la suite 1, 2, ... n. On fait la somme de ces p nombres. Evaluer la somme de toutes ces sommes partielles. 79

11450. (Professor Bernès.)—Dans un triangle ABC, les perpendiculaires en B, C, à AB, AC, rencontrent AC, AB en D, E. Démontrer que, si une parallèle quelconque à BC rencontre AB, AC en F, G, l'axe radical des circonférences BGD, CFE passe par A et par le centre de la circonférence ABC. 57

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11452. (Professor De Longchamps.)—Resoudre les equations $(y-z+b^2-c^2)^2 = 4a^2x$, $(z-x+c^2-a^2)^2 = 4b^2y$, $(x-y+a^2-b^2)^2 = 4c^2z$. 97

11458. (Editor.)—If P be any point in the circumference, and BC, CD two equal arcs, of a semicircle APB; then, if CA, PB meet in E, and AD, PC in F, prove that AD is perpendicular to EF. 54

11465. (H. Brocard.)—On donne deux droites rectangulaires AB et CD, non situées dans un même plan; soit AC leur plus courte distance. Une droite s'appuie sur AB et CD en interceptant des segments égaux à partir de A et C. Déterminer la courbe d'intersection de la surface engendrée par cette droite avec la sphère décrite sur AC comme diamètre. 55

11484. (Professor Orchard, M.A., B.Sc.)—What relation must hold among the coefficients in the general cubic equation $x^3+px^2+qx+r=0$, in order that it may be represented as the product of two factors, one simple and one quadratic? 112

11485. (Professor Zerr.)—1000 c.c. of hydrogen precipitates 9·65 grams of silver from a solution of silver acetate; also, by heating platinum tetrachloride in 1000 c.c. of hydrogen, and precipitating the hydrogen chloride with the silver nitrate formed from the silver above found, I have 4·4061 grams of platinum left, when 12·8263 grams of silver chloride are formed. What is the atomic weight of silver and platinum? ... 77

11486. (The late Professor Wolstenholme, Sc.D.)—A rod AB is marked at random in P, and points Q, R are then taken at random in AP, PB, respectively; prove that (1) the chance that the sum of the squares on AQ, RB will exceed the sum of those on QP, PR is $\frac{1}{2}$; but (2), when Q, R are first taken at random in AB, and P then taken at random in QR, the chance of the same event is $\frac{3}{8}(3-2\log 2)$. 64

11489. (Professor Balitrand.)—Soit $a^2/x^2+b^2/y^2-1=0$ l'équation d'une kreuzcurve. La droite qui joint les projections A, B d'un point N de cette courbe, sur les axes, touche l'ellipse qui correspond à l'équation $x^2/a^2+y^2/b^2-1=0$ en un point M. Démontrer que la parabole tangente aux axes, aux points où ils sont coupés par la tangente en N à la kreuzcurve, passe au point M, tangentiellement à l'ellipse..... 30

11491. (Professor Shields.)—A. owned a tract of land in the form of an equilateral triangle. He had three sons, B., C., D.; three daughters, E., F., G.; and three grandchildren, H., I., J. Of this land, he gave to each son one of the three equal and largest possible circular tracts; to each daughter, one of the corner sections cut off by one of these circular tracts; and to each grandchild, one of the side sections cut off by two of these circular tracts; and for himself he reserved the tricuspid section between the three circular tracts. This section contained $(2500\sqrt{3}-3927)$ sq. rds., or 403·125+ sq. rds. Find the share of each son, daughter, and grandchild. 108

11495. (Professor Aiyar.)—Three points are taken at random on the circumference of a circle: prove that the probability of their lying on the same semicircle is $\frac{3}{4}$. 32

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11496. (Professor Mukhopadhyay.)—Given the lengths of two pairs of opposite edges of a tetrahedron, and the sum of the lengths of the remaining pair, prove that the volume will be a maximum when the angles between the planes meeting in these two edges respectively are equal. 29

11505. (D. Biddle.)—A marble falls from a given height h above a horizontal plane, but is intercepted at $\frac{1}{2}h$ by a small inclined plane, from which it rebounds. The angle of this small plane is constantly varying between 0° and 90°. Find the mean distance from the vertical at which the marble strikes the horizontal plane. 65

11512. (Professor Sylvester.)—Find all the solutions of the equation $x^z - y^t = 1$, where z, t are integer and x, y prime numbers. 25

11513. (Professor Crofton, F.R.S.)—If Σ, Σ' be the two segments into which the chord C divides the area Ω, prove that

$$\iint C^4 dp\, d\omega = 6 \iint \Sigma\Sigma' dp\, d\omega. \quad \ldots\ldots \ldots\ldots \quad 49$$

11515. (Professor Zerr.)—A pack of fifty-two cards contains twenty 10-spots, four 9-spots, four 8-spots, four 7-spots, four 6-spots, four 5-spots, four 4-spots, four 3-spots, and four 2-spots. The cards are well mixed, and then one card is taken and the number of spots noted, and enough cards (regardless of spots) are placed upon this card to count up to twelve. If the card first taken was a 7-spot, then count that card 7, the next 8, the next 9, and so on to twelve. This operation is repeated until all the cards are taken, except those remaining which are not enough to count up to twelve. Four are subtracted from the number of piles obtained, the remainder multiplied by 13, to this product is added the number of cards remaining, and the sum is equal to the number of spots on the bottom cards of all the packs. Required investigation. 25

11516. (Professor Orchard, M.A., B.Sc.)—Show how the earth's radius may be found by sailing along a great circle, and taking latitudes. 71

11517. (Professor Duran Loriga.)—Demostrar que si un triángulo isósceles tiene el ángulo en el vértice de 45°, la base es media proporcional entre el cuádruplo de la altura y el exceso de la altura sobre la base..... 28

11519. (Professor Farjon.)—On détermine le plan radical des sphères ayant pour diamètres deux arêtes opposées d'un tétraèdre; les trois plans ainsi obtenus se coupent en un certain point. Démontrer que ce point et le centre de la sphère circonscrite au tétraèdre sont symétriques par rapport au centre de gravité du tétraèdre. 94

11520. (Professor Neuberg.)—Trouver

$$\int \frac{\sin 3x}{\cos 4x}\, dx. \quad \ldots\ldots \quad 76$$

11525. (Professor Shields.)—Two boats, A and B, a certain distance apart on a river, whose current rate is $2\frac{1}{2}$ miles an hour, agreed to run

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to each others' starting-points. Boat A, whose running rate in still water is 14 miles per hour, is 10 hours longer in running *up stream* from B to A than in running down the same distance from A to B; and boat B, whose running rate in still water is 8 miles an hour, is 41$\frac{1}{7}$ minutes longer in running 8 miles *up* the river, than in running 8 miles *down* the same. They both start at the same time from their respective places (A 14 —— 2$\frac{1}{2}$ —— B 8), and run to each others' starting-points (the fastest boat, A, running down stream, while B runs up). Find (1) the entire distance run by the boats, and (2) how far each boat was from its starting-point when they met or passed. 95

11527. (Professor Thiry.)—Construire le triangle rectangle dont on connaît la hauteur relative à l'hypoténuse, ainsi que la distance du sommet de l'angle droit au pied de la bissectrice d'un angle aigu. 50

11533. (I. Arnold.)—Construct a right-angled triangle, having given the median lines from the ends of the hypotenuse to the opposite sides. 51

11535. (H. Brocard.) — Au point P d'une courbe Δ on mène l'ordonnée PB et la tangente PA rencontrant l'axe OX en A; la perpendiculaire élevée en A sur PA coupe PB en M. Déterminer la courbe Δ par la condition que la courbe (M) soit symétrique de Δ par rapport à l'origine..... 76

11536. (R. Knowles, B.A.)—A circle touches a given parabola S at a point P at an end of the latus-rectum, and cuts it in QR; PC is the chord of curvature at P; K the pole of QR with respect to S; QR meets the tangent at P in M. Prove that (1) the envelope of MK is a parabola S′, which touches S at P and cuts it in Q′R′; (2) C is the pole of Q′R′ with respect to S′; (3) if K′ is the pole of Q′R′ with respect to S, then Q′R′, the axis of S′, and the perpendicular from K′ on the axis of S are concurrent..... 126

11538. (R. Tucker, M.A.)—The image of a parabola with respect to the tangent at P $(am^{-2}, 2am^{-1})$ is the parabola

$$\left[m(m^2-1)Y + 2m^2X + 2a\right]^2 + 4am^2(m^2+1)\left[X(m^2-1) - 2mY + 2a\right] = 0.$$

Find the locus of the vertices and the envelope of the tangents at them for different positions of P. 63

11540. (J. H. Grace.)—E and F are the mid-points of the diagonals (internal) of a quadrilateral. The line EF meets the third diagonal in G, and the sides of the quadrilateral in P, Q, R, and S, respectively (P and R being on opposite sides). Prove that

$$GE \cdot GF = GP \cdot GR = GQ \cdot GS. \text{ ..... } 66$$

11545. (Professor Zerr.)—At the corners of a square, of side d, there are four towers of heights $a, b, c, (a^2+c^2-b^2)^{\frac{1}{2}}$. Find the length of a ladder and the distance of the point from the foot of each tower, at which, if the ladder be placed, it will just reach the top of each tower..... 86

11548. (Professor Hudson, M.A.)—From two points in a plane two series of straight lines are drawn at the same equal small angular intervals, dividing the plane into quadrilateral elements; find (1) the angle between the diagonals of these elements ultimately when the angular intervals are indefinitely small, and (2) the locus of points such that this angle is a right angle. 83

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11549. (Professor W. H. S. Monck, M.A.)—Supposing the chance of a Knight occupying a particular square to be proportional to the number of squares from which that square might have been reached, find the mean range of the Knight. 57

11550. (Professor Ramaswami Aiyar, M.A.)—Prove that, if triangles inscribed in a circle X may be circumscribed to a given conic, the locus of their orthocentres is a circle Y; and triangles may be inscribed in Y which circumscribe the conic; the locus of *their* orthocentres is X.... 104

11551. (Professor Bhattacharya.)—From an unknown number of balls each equally likely to be red, white, or blue, ten are drawn, and turn out to be five red, three white, and two blue; prove that if three more balls be drawn the chance of their being one red, one white, and one blue is $\frac{7.8}{4.5.6}$; the chance of three red is $\frac{5}{6.6}$; of three white is $\frac{4}{9.1}$; and of three blue is $\frac{2}{9.1}$; and the chance of there being no white ball in the three is $\frac{3.3}{9.7}$. 90

11552. (Professor Nilkantha Sarkar.)—Prove that

$$\int_0^\infty \frac{x^4 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2} \frac{bc+ca+ab}{(b+c)(c+a)(a+b)}. \quad \ldots\ldots 75$$

11559. (Professor Bernès.)—Dans l'angle A d'un triangle ABC et dans son opposé par le sommet, on trace deux droites DE, D'E' antiparallèles à BC et équidistantes de A. Faire voir que, si M est la rencontre de BE et CD, et M' la rencontre de BE', CD', les droites AM, AM' sont conjuguées harmoniques relativement à la médiane et à la symédiane issues de A dans le triangle ABC. 83

11561. (Professor Rebuffel.)—Les arêtes AB, AD, AA' et la diagonale AC' d'un parallélipipède ABCD, A'B'C'D' sont rencontrées par une sphère qui passe par le sommet A aux points β, δ, α', γ'. Prouver que

$$(1)\ AB.A\beta + AD.A\delta + AA'.A\alpha' = AC'.A\gamma',$$

et (2) que le lieu des centres des sphères qui passent par le sommet O d'un angle trièdre, et qui rencontrent les arêtes en des points A, B, C tels que $OA+OB+OC = l$, est un plan. 52

11562. (Professor Malet, F.R.S.)—If a right line be drawn cutting a conic U in A and B, and a conic V in C and D, prove that the tangents to U at A and B intersect the tangents to V at C and D in four points which lie on a conic passing through the four points of intersection of U and V. 67

11563. (Professor Lange, Ph.D.)—(1) Let A_0, B_0, C_0 be the midpoints of the sides BC, CA, AB of the triangle ABC, J the centre, and A_1, B_1, C_1 the points of contact of the inscribed circle with the sides. Turn the triangle round each of the lines AJ, BJ, CJ so as to have it successively in the positions AB_aC_a, BA_bC_b, CA_cB_c, while A_1, B_1, C_1 coincide with the points A_2, B_2, C_2 of the inscribed circle. Then A_0A_2, B_0B_2, C_0C_2 will meet in one point T, and this point proves to be the exterior centre of similitude of the inscribed circle J, and Feuerbach's circle F, and to be a common point of them. Hence J and F touch in T. (2) Let BB_a, CC_b meet in P_a; AA_c, CC_a in P_b; AA_b, BB_a in P_c; AP_a, BC in X; BP_b, AC in Y; CP_c, AB in Z; then X, Y, Z lie in a straight line, viz., the common tangent of F and J in T..... 69

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11566. (Belle Easton, B.Sc.)—If points A, B are taken within a circle of radius r whose centre is C, prove that the diameters of the circles which can be drawn through A and B to touch the given circle, are (the symbols referring to the parts of the triangle ABC) the roots of $x^2(r^2c^2 - a^2b^2\sin^2 C) - 2xrc^2(r^2 - ab\cos C) + c^2(r^4 - 2r^2ab\cos C + a^2b^2) = 0$. 90

11570. (E. M. Langley, M.A.)—AD, CF are the external common tangents to two circles ABC, DEF; BE one of the internal. The diameters through B and E cut AC and DF respectively in G and H. Show (1) that GH bisects BE; and find (2) whether the above theorem is also true if "external" and "internal" be interchanged. 59

11572. (Morgan Brierley.) — P is a variable point external to a circle given in magnitude and position, and PD a line making a given angle with a diameter of the circle produced; find the *locus* of P, when $PD^2 + PC^2$ is a given magnitude, PC being a tangent to the circle. ... 77

11574. (I. Arnold.)—A uniform circular plate rests with its centre O on a fixed point. Three weights P, Q, R are attached to the circumference so as to keep it in equilibrium in a horizontal position; it is required (1) to determine the angles POQ, QOR, ROP in terms of the weights, and without assuming any given diameter; (2) assuming the diameter 20 and the weights P, Q, R, 20, 30, 40, find the sides of the triangle made by the weights, or their points of suspension. 123

11575. (J. C. St. Clair.)—If P be the point of concurrence of the lines joining the vertices of a triangle ABC to the points of contact of any inscribed conic with the opposite sides, prove that the tangents at A, B, C to any conic passing through A, B, C, P form a triangle which is self-conjugate with respect to the first conic. 81

11576. (W. J. Greenstreet, M.A.)—A parabola always touches the rectangular axes so that the chord of contact is a constant (a); show that the locus of its focus is a curve $(x^2 + y^2)^3 = a^2x^2y^2$. 87

11578. (A. J. Pressland, M.A.) — If k be the side of the square inscribed in a triangle ABC, having one side collinear with BC, and p be the length of the perpendicular from A on BC, prove that

$$R^{-1} = a^{-1} + p^{-1}. \quad \ldots\ldots 75$$

11591. (Professor Lucas.)—De combien de manières un pion du jeu de dame, placé en un coin du damier, peut-il se rendre sur le bord opposé, en progressant par cases consécutives parallèlement à l'une ou l'autre diagonale? Le damier a $2n$ ou $2n+1$ cases par côté. 70

11598. (W. J. Greenstreet, M.A.)—Through any point on the axis of homology of a triangle ABC and its pedal, parallels are drawn to AB, BC, CA. The six points where these parallels meet the sides lie on a conic. 106

11599 and 11670. (R. Tucker, M.A.)—11599. In a triangle ABC, points are taken on the sides such that

$$BP : CP = CQ : AQ = AR : BR = m : n = CP' : BP' = AQ' : CQ' = BR' : AR';$$

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prove that (1) radical axis of circles PQR, P'Q'R' passes through the centroid and "S." points of ABC; (2) if QR, Q'R' cut in 1, RP, R'P, in 2, PQ, P'Q' in 3, then the equation to the circle 123 is

$$abc\Sigma a\beta\gamma = mn\Sigma aa \cdot \Sigma aa \left\{ -mna^2 + (m^2 + mn + n^2)(b^2 + c^2) \right\};$$

also (3) indicate other properties of the system. 119

11603. (Rev. T. R. Terry, M.A.)—Three persons, whose skills are in the ratio $a : b : c$, play a match, the winner of which is to be the one who first wins two consecutive games. The winner of any game plays the remaining player, the first pair being chosen by lot. Show that the chance that the person who does not play in the first game wins the match is $\frac{1}{3}\frac{1-2x}{1-x}$, where $x(b+c)(c+a)(a+b) = abc$. 92

11606. (A. Kahn, B.A.)—Two circles intersect in A, B. Through A any line CAD is drawn intersecting the circles in C and D; and through B the line EBF is drawn parallel to CD, and cutting the circles in E and F. EC and BA are produced to meet in G, and FD and BA are produced to meet in H. Prove that the square of the tangent from G to either circle is to the square of the tangent from H as $EA^2 - AB^2$ is to $FA^2 - AB^2$. 123

11611. (Morgan Brierley.)—If X_1, X_2, X_3 denote the three perpendiculars from the three angles of a triangle ABC upon the opposite sides, and Y_1, Y_2, Y_3 perpendiculars upon the same from a given point within the triangle, prove that $\frac{Y_1}{X_1} + \frac{Y_2}{X_2} + \frac{Y_3}{X_3} = 1$. 112

11612. (A. E. Jolliffe.)—A particle describes freely a curve under the action of a force towards a point O. Show that the reciprocal of this curve with respect to O can be described freely by a particle under the action of a force, towards O, proportional to the reciprocal of the normal force in the first orbit. 126

11616. (Professor Minchin, M.A.)—Of five coplanar forces in equilibrium, given the lines of action of all, the magnitude of one, and the ratio of two others, find the magnitudes of all by a very simple geometrical construction. 98

11625. (Professor de Longchamps.)—Soient MA, MB deux tangentes à une parabole P. Le demi-paramètre p de cette courbe est donné par la formule $p \cdot MO^3 = (\text{aire } MAB)^2$, O désignant le milieu de AB. En utilisant cette relation, montrer que le lieu des foyers des paraboles de forme invariable, inscrites dans un angle fixe θ, est la quartique correspondant à l'équation $4x^2y^2 \sin^4\theta = p^2(x^2 + y^2 + 2xy\cos\theta)$. 100

11630. (Professor de Wachter.)—Un cône circulaire droit, formé d'une matière homogène de densité d, est entièrement immergé dans un vase contenant deux liquides superposés de densités δ_1 et δ_2 ($\delta_1 > \delta_2$). Montrer que le plan de séparation des deux liquides divise l'axe du cône dans un rapport égal à $\left(\frac{\delta_1 - \delta_2}{d - \delta_2}\right)^{\frac{1}{3}} - 1$. 98

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11633. (D. T. Griffiths.)—S and S′ are the foci of an oval of Cassini, and P is any point on the curve. If the normal at P meets SS′ at G, prove that $SG : S'G = SP^2 : S'P^2$..... 101

11648. (Professor Sylvester.)—Show that the complete integrals of the equations in differences $\frac{u_{x+1}-3u_x}{u_x-3u_{x-1}}=\frac{u_x^3}{u_{x-1}^3}$, $\frac{v_{x+1}+3v_x}{v_x+3v_{x-1}}=\frac{v_x^3}{v_{x-1}^3}$, is

$$u_x = C(a^{3^x}-a^{-3^x}), \quad v_x = C(a^{3^x}+a^{-3^x}).$$..... 97

11649. (Professor Lampe, Ph.D. Suggested by Quest. 4269, Vol. LIII., p. 131.) — A circle being divided into an *odd* number n, $= 2m+1$, of equal parts, let $A_1, A_2, A_3, \ldots, A_n$ be the successive points of division, O an arbitrary point of the circumference situated between A_1 and A_n. Join O to $A_1, A_2, A_3, \ldots, A_n$ by the chords $a_1, a_2, a_3, \ldots, a_n$; then prove that

(1) $(a_1+a_n)(a_2+a_{n-1})(a_3+a_{n-2})\ldots(a_{\frac{1}{2}(n-1)}+a_{\frac{1}{2}(n+3)}) = a_{\frac{1}{2}(n+1)}^{\frac{1}{2}(n-1)}$;

(2) $a_1+a_n+a_3+a_{n-2}+a_5+a_{n-4}+\ldots-(a_2+a_{n-1}+a_4+a_{n-3}+a_6+a_{n-5}+\ldots)$
$= (-1)^{m-1}a_{\frac{1}{2}(n+1)}$;

and find (3) how these relations should be modified if n is an *even* number. 115

11666. (The Editor.)—Construct a triangle, having given the orthocentre P, the point Q where the circumcircle is cut by the external bisector of the vertical angle A, the angle AQP, and the difference of the angles B, C, at the base of the triangle. 99

11667. (Ven. Archdeacon Wilson, M.A.)—When $4n+1$ is a prime number, it is an old and well-known property of numbers that it is expressible in the form of the sum of two squares. But the proofs throw little or no light on "the reason why." Can any connexion be shown, or any explanation given of this curious property? [Reference may be made to FERMAT, who announces the theorem in his *Observations on Diophantus*, VII.; to EULER, by whom it is proved in the *Nov. Com. Petr.*, V., p. 3; to SERRET, by whom it is proved in Art. 294 of his *Alg. Supérieure* (3rd edit., 1866); or to ROUSE BALL'S *Math. Recreations*.] 113

11670. (R. Tucker, M.A.)—In Question 11599, (3) prove the conic about P′PQ′QR′R to be concentric, similar, and similarly situated with the minimum circum-ellipse of ABC; (4) the "S." points of PQR, P′Q′R′ are

$$\frac{aa}{mq^2+nr^2}=\ldots=\ldots; \quad \frac{aa}{nq'^2+mr'^2}=\ldots=\ldots;$$

where $p = QR, \ldots; p' = Q'R', \ldots$; and the "S." point of 123 is

$$aa/\left[(m^2+n^2)a^2+mn(b^2+c^2)\right]=\ldots=\ldots.$$ 119

11671. (H. Brocard.) — Soient AA′, BB′ deux diamètres rectangulaires d'un cercle O. Une sécante quelconque menée par A rencontre BB′ en C; la perpendiculaire élevée en C sur AC coupe

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AA′ en D ; trouver le lieu géométrique du point M, intersection de AC avec BD. En déduire la construction d'un angle dont la tangente soit le carré de la tangente d'un angle donné. 124

11676. (J. H. Grace.)—If Q and Q_1 are the centres of curvature of the extremities of a focal chord of a parabola whose focus is S, and if a line parallel to that focal chord meet SQ, SQ_1 in C and D, and the axis of the parabola in B, then SB = CD. 125

11681. (D. Biddle.)—A hollow cylinder, of length l, and intermediate radius r, revolves on its axis with an angular velocity v. A particle is proceeding within it with a uniform velocity V, parallel to the axis, but otherwise at random. In the plane of the axis, and extending to the circumference, is a partition. Find the probability that the particle and the partition will collide, (1) when the partition extends right across the cylinder, (2) when it is on one side of the axis only. Also give the mean distance at which the collision occurs in each case..... 101

11697. (Professor Bernes.)—DE étant une corde de la circonférence circonscrite à un triangle ABC; F, G les points de rencontre de DB, DC avec une sécante variable AZ; H, K les points où les parallèles à DE par F, G rencontrent AE ; quel est le lieu de l'intersection M des droites BH, CK, lorsque AZ tourne autour de A ? 124

11703. (Editor.)—If three circles A, B, C touch each other two and two, and a common tangent to A and B is parallel to a common tangent of A and C, and if a, b, c be the radii, and p, q the distances of the centres of B and C from that diameter of A which is normal to the two parallel tangents, prove that $pq = 2a^2 = 8bc$. 113

11704. (D. Biddle.)— Over the wheel of an Attwood's machine for illustrating the laws of gravitation passes a rope, to one end of which is attached a weight A, and to the other a pulley B, over which play weights a and b. Find the positions of B and b at the end of each successive second, disregarding friction and the resistance of the air.

[This, in a general form, is the question of which Quest. 11389 was a particular instance.] 113

APPENDIX.

Unsolved Questions 129

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

11512. (Professor SYLVESTER.)—Find all the solutions of the equation $x^z - y^t = 1$, where z, t are integer and x, y prime numbers.

Solution by H. W. CURJEL, B.A.

If $x^z - y^t = 1$, where t and z are whole numbers and x and y are primes, $x = 2$ or $y = 2$.

If $y = 2$, $x_z = 2^t + 1$; z must $= 1$ or 2, for $x - 1 = 2^u$, where u is a whole number; therefore $2^t = (2^u + 1)^z - 1$.

$z = 2$ gives $x = 3$, $t = 3$. $z = 1$ gives $x = 2^t + 1$, where t obviously must be a power of $2 = 2^n$, *i.e.*, $x = 2^{2^n} + 1$.

If $x = 2$, $y^t = 2^z - 1$; z evidently must be a prime, for

$$2^{qr} - 1 = (2^q - 1)(2^{(r-1)q} + 2^{(r-2)q} + \ldots + 2^{2q} + 2^q + 1)$$
$$= (2^r - 1)(2^{(q-1)r} + 2^{(q-2)r} + \ldots + 2^{2r} + 2^q + 1),$$

which obviously cannot be a power of a prime. t appears to be always $= 1$; $\therefore$ $y = 2^z - 1$, where z is a prime.

Hence we see that z is always a prime, and $t = 1$ or a power of 2, and x or $y = 2$.

11515. (Professor ZERR.)—A pack of fifty-two cards contains twenty 10-spots, four 9-spots, four 8-spots, four 7-spots, four 6-spots, four 5-spots, four 4-spots, four 3-spots, and four 2-spots. The cards are well mixed, and then one card is taken and the number of spots noted, and enough cards (regardless of spots) are placed upon this card to count up to twelve. If the card first taken was a 7-spot, then count that card 7, the next 8, and so on to twelve. This operation is repeated until all the cards are taken, except those remaining which are not enough to count up to twelve. Four are subtracted from the number of

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piles obtained, the remainder multiplied by 13, to this product is added the number of cards remaining, and the sum is equal to the number of spots on the bottom cards of all the packs. Required investigation.

Solution by J. C. St. Clair.

In general, let N be the number of cards employed; n the number to which each pile is made up; p the number of piles; r the number of remaining cards; and S the sum of the spots on the bottom cards.

It is obvious that, whatever may be the composition of the pack, the number of cards in each pile is $n+1$ less the number of spots at bottom.

Hence $N-r=p(n+1)-S$, and $S=p(n+1)-N+r$.

If, as in the question, $N=52$, $n=13$, the formula becomes

$$S=13p-52+r=13(p-4)+r.$$

[If the first card taken is a 2-spot, the number of cards necessary to make twelve is 11, if the next card is a 4-spot, the number necessary is 9, if the next card is a 3-spot, the number necessary is 10, if the next is a 7-spot, the number necessary is 6, if the next is a 2-spot, the number necessary is 11, if the next is a 10-spot, the number necessary is 3; in all, six piles and two cards remaining, $(6-4)\,13+2=28$; also $2+4+3+7+2+10=28$.

The problem occurs on pp. 69–71 of *Problèmes Plaisants et Délectables*, by Bachet (1612–24), republished by Gauthier-Villars, Paris, 1879.]

Mr. Greenstreet gives the following solution of the problem:—

If x is the spot-number of the bottom card of a pile, there must be $13-x$ cards in that pile. If there are 52 cards in the pack, this must equal the sum of the numbers of cards in each pile, together with the r remaining cards; that is

$\Sigma(13-x_p)+r=52$, or $13p-\Sigma x_p+r=52$, or $\Sigma x_p=13(p-4)+r$.

If n cards are taken, and the sum of spots on bottom card, and the cards placed on it, be m, we have $\Sigma x_p=(m+1)p+r-n$.

In an *écarté* pack, we have $n=32$, and it is best to take $m=15$.]

11165. (The late Professor Clifford, F.R.S.)—A rigid body is made by fixing the flat side of a hemisphere to one of the flat ends of a portion of a circular cylinder of equal radius cut out by planes perpendicular to the axis, the whole being of uniform density; find (1) the time of small oscillations about a diameter of the other flat, circular end; and, if the body rotates about an axis through the centre of the hemisphere, making an angle θ with the axis of the cylinder, determine (2) the moment of momentum.

Solution by H. W. Curjel, B.A.

1. Let a be the radius of the cylinder, and b its length, and k the radius of gyration of the whole body about the axis of suspension, and h

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the distance of the centre of gravity of the whole below the axis of suspension; then

$$h = \frac{6b^2 + 8ab + 3a^2}{2(3a + 2a)}, \quad k^2 = \frac{16a^3 + 33a^2b + 24ab^2 + 5b^3}{20(3b + 2a)}.$$

Equation of motion is $Mk^2\theta = -Mgh\sin\theta$;

hence time of a small oscillation

$$= 2\pi\left(\frac{k^2}{gh}\right)^{\frac{1}{2}} = 2\pi\left(\frac{16a^3 + 33a^2b + 24ab^2 + 5b^3}{g(6b^2 + 8ab + 3b^2)}\right)^{\frac{1}{2}}.$$

2. Let ρ be the density of the body, and ω the angular velocity, then moment of momentum = $\omega \times$ moment of inertia about axis of rotation,

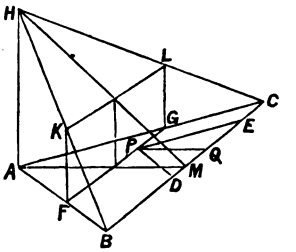
$$= \omega\pi\rho a^2\left\{\tfrac{1}{4}a^2b\,(2\cos\theta + \sin\theta) + \tfrac{1}{12}b^3\sin\theta + a^3\left[\tfrac{4}{15}(\cos\theta + \sin\theta) - \tfrac{9}{14}\sin\theta\right]\right\}$$

$$= \omega\pi\rho a^2\left\{\tfrac{1}{12}b^3\sin\theta + \tfrac{1}{4}a^2b\,(2\cos\theta + \sin\theta) + a^3\left(\tfrac{4}{15}\cos\theta + \tfrac{12}{840}\sin\theta\right)\right\}.$$

5132. (Professor Crofton, F.R.S.) — From any point P within a triangle lines are drawn parallel to the two sides, meeting the base on D, E. Show (1) [as may be done without integration] that the mean value of the triangle PDE is one-sixth of the former triangle; and find (2) the mean value of the segment DE.

Solution by Professor Schoute.

1. If P moves on the line FG parallel to the base BC of the triangle ABC, the surface of triangle PDE stands to that of ABC as PQ^2 to AM^2. Now we describe on FG in the plane through FG normal on ABC the rectangle FGLK, the height of which is equal to PQ. In this manner a tetrahedron ABCH is generated, the height of which is equal to AM. Now the mean value of the quotient



$$\frac{\Delta PDE}{\Delta ABC}, \text{ or } \left(\Sigma FG\frac{PQ^2}{AM^2}\right)\Big/\Sigma FG,$$

is evidently equal to the moment of the tetrahedron with respect to the plane through BC normal to ABC, divided by the product $AM^2\,\Delta ABC$, or $\frac{1}{6}$.

2. In the same manner the mean value of

$$\frac{DE}{BC}, \text{ or } \left(\Sigma FG\frac{PQ}{AM}\right)\Big/\Sigma FG,$$

is the volume of the tetrahedron divided by the product $AM\Delta ABC$, or $\frac{1}{3}$.

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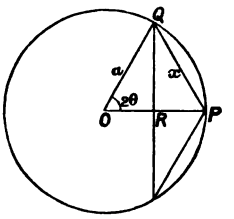
6016. (The Editor.)—Find the probability that a circle drawn at random on the surface of a sphere will cut off a portion of spheric surface less than one-fourth of the surface of the sphere.

Solution by H. J. Woodall.



Suppose that any linear radius $= x$ (see figure) be equally likely, $\frac{1}{2}x = a\sin\theta$.

Area of circle $= \mathrm{PR}/2a \times$ surface $= \frac{1}{4}x^2/a^2\,\mathrm{S}$.

Therefore ratio $= \frac{1}{4}x^2/a^2$,

$$\text{prob.} = \int_0^a \frac{x^2}{4a^2}\,dx \Big/ \int_0^{2a} \frac{x^2}{4a^2}\,dx = \frac{1}{8}.$$

This is what would happen in the case of an ordinary compass with legs. If the circle is traced by means of a piece of thread stretched over the surface,

$$\mathrm{PQ\,(arc)} = 2\theta a = y \text{ say}, \quad \mathrm{PR} = \tfrac{1}{2}x^2/a = 2a\sin^2\theta.$$

Hence the function we have to consider is

$$2a\sin^2(y/2a) = a\left\{1-\cos(y/a)\right\}.$$

When area of circle $= \frac{1}{4}$ area of surface,

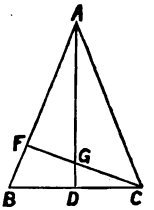
$$\mathrm{PR} = \tfrac{1}{2}a; \quad \therefore \sin\theta = \tfrac{1}{2}, \quad \theta = \tfrac{1}{6}\pi, \quad y = \tfrac{1}{3}a\pi,$$

$$\text{prob.} = \frac{\int_0^{\frac{1}{3}a\pi} a\left\{1-\cos y/a\right\}dy}{\int_0^{a\pi} a\left\{1-\cos y/a\right\}dy} = \frac{\left[y-a\sin y/a\right]_0^{\frac{1}{3}a\pi}}{\left[y-a\sin y/a\right]_0^{a\pi}} = \frac{2}{3}-\frac{\sqrt{3}}{2\pi}.$$

Solutions may be given on other hypotheses as to random takings. The above solution rests on the assumption that the part cut off includes the pole of the circle.

11517. (Professor Duran Loriga.)—Demostrar que si un triángulo isósceles tiene el ángulo en el vértice de 45°, la base es media proporcional entre el cuádruplo de la altura y el exceso de la altura sobre la base.

Solution by H. W. Curjel, B.A.; Professor Zerr, M.A.; *and others.*



Let ABC be the triangle having $\angle\mathrm{BAC} = 45°$.
Draw AD perpendicular to BC.
Draw CF perpendicular to AB meeting AD in G.
Then triangles AFG, CBF are equiangular, and

have $\mathrm{FC} = \mathrm{AF}$;

therefore $\mathrm{AG} = \mathrm{BC}$;

also triangles ADC, CDG are similar; therefore

$$\mathrm{GD} : \mathrm{DC} = \mathrm{DC} : \mathrm{AD};$$

therefore $\mathrm{GD} : \mathrm{BC} = \mathrm{BC} : 4\mathrm{AD}$.

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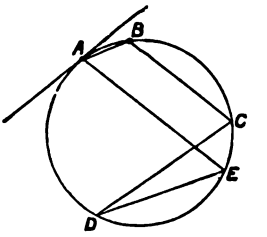
5322. (J. L. McKenzie, B.A.)—A triangle ABC is inscribed in a conic; through C a line is drawn parallel to the tangent at A, and meeting the conic in D; through A a line is drawn parallel to BC, and meeting the conic in E: prove that DE is parallel to AB.

Solution by Professor Schoute.

We consider the inscribed hexagon AABCDE. According to Pascal's theorem, the three pairs AA and CD, AB and DE, BC and AE intersect in three collinear points. But the join of the first and the third point is the line at infinity. Therefore the point common to AB and DE is at infinity, &c. [For another solution, see Vol. lvii., p. 52.]



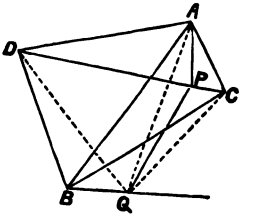
11496. (Professor Mukhopadhyay.)—Given the lengths of two pairs of opposite edges of a tetrahedron, and the sum of the lengths of the remaining pair, prove that the volume will be a maximum when the angles between the planes meeting in these two edges respectively are equal.

Solution by Professors Aiyar, Chakravarti, *and others.*

Let ABCD be a tetrahedron of volume V, in which the sum of the edges AB, CD is given, as well as the other edges. We have to show that V is a maximum when the dihedral angles (AB), (CD) are equal.

Let ∂, V be the differential of the volume when [AB] increases by ∂[AB] in consequence of the angle (CD) increasing by ∂(CD) (the other sides remaining constant), we shall find a relation between ∂, V and ∂[AB].

Draw AP perpendicular to CD, and PQ perpendicular to CD, in the plane BCD, meeting the parallel to CD drawn through B in Q.



Now, volume of ABCD = volume of AQCD; therefore

$$V = \tfrac{1}{6} \cdot CD \cdot AP \cdot PQ \cdot \sin(CD);$$

therefore $$\partial_1 V = \tfrac{1}{6} \cdot CD \cdot AP \cdot PQ \cdot \cos(CD) \cdot \partial(CD) \quad \ldots\ldots(1).$$

Again, $$AB^2 = BQ^2 + QA^2 = BQ^2 + QP^2 + PA^2 - 2 \cdot AP \cdot PQ \cdot \cos(CD).$$

Differentiating, $$AB \cdot \partial[AB] = AP \cdot PQ \cdot \sin(CD) \cdot \partial(CD) \quad \ldots\ldots(2).$$

By (1) and (2), $$\partial_1 V \tan(CD) = \tfrac{1}{6} \cdot AB \cdot CD \cdot \partial[AB] \quad \ldots\ldots(3).$$

Similarly, if $\partial_2 V$ be the differential of V when [CD] alone varies, we have $$\partial_2 V \tan(AB) = \tfrac{1}{6} \cdot AB \cdot CD \cdot \partial[CD] \quad \ldots\ldots(4).$$

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Now, when both AB and CD vary, if ∂V be the differential of V, we have $\partial_1 V + \partial_2 V = \partial V = 0$, when V is a maximum.....(5).
But, by addition of (3) and (4), we see, if AB + CD be constant, that

$$\partial_1 V \tan (CD) + \partial_2 V \tan (AB) = 0 \quad \text{.....(6).}$$

From (5) and (6) we see at once that

$$\tan (AB) = \tan (CD);$$

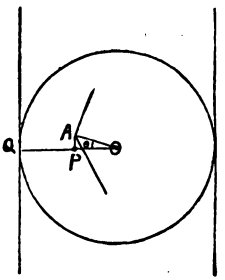
which proves the theorem.

1274. (The EDITOR.)—If an indefinite number of parallel equidistant lines is drawn on a plane, and a regular polygon, the diameter of whose circumscribed circle is less than the distance between consecutive parallels, is thrown at random on the plane, prove that the probability that the polygon will fall on one of the lines is l/l', where l is the perimeter of the polygon, and l' the circumference of the greatest circle that can be placed between the parallels.

Solution by H. J. WOODALL.

Draw the greatest circle concentric with the polygon;

$$\text{prob.} = \int \text{OP} d\theta \Big/ \int \text{OQ} d\theta = \int a \cos\theta \, d\theta \Big/ \int r \, d\theta.$$



These integrals must be taken between the limits 0 and π/n; whence

$$\text{probability} = 2a \,.\, n \sin(\pi/n) / 2\pi r = l/l'.$$

But all this follows immediately from Prof. CROFTON's Theory (see *Phil. Trans.*, 1868).

11489. (Professor BALITRAND.)—Soit $a^2/x^2 + b^2/y^2 - 1 = 0$ l'équation d'une kreuzcurve. La droite qui joint les projections A, B d'un point N de cette courbe, sur les axes, touche l'ellipse qui correspond à l'équation $x^2/a^2 + y^2/b^2 - 1 = 0$ en un point M. Démontrer que la parabole tangente aux axes, aux points où ils sont coupés par la tangente en N à la kreuzcurve, passe au point M, tangentiellement à l'ellipse.

Solution by H. W. CURJEL, B.A.; W. J. GREENSTREET, M.A.; *and others.*

Let coordinates of M, N be $(a\cos\theta, b\sin\theta)$, $(a\sec\theta, b\operatorname{cosec}\theta)$; then tangent at N is $a\sin^3\theta\,(y - b\operatorname{cosec}\theta) + b\cos^3\theta\,(x - a\sec\theta) = 0$.

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This cuts the axes at points $(a\sec^3\theta, 0)$, $(0, b\operatorname{cosec}^3\theta)$. Parabola touching the axes at these points is

$$\left(\frac{x\cos^3\theta}{a}\right)^{\frac{1}{3}} + \left(\frac{y\sin^3\theta}{b}\right)^{\frac{1}{3}} = 1.$$

This evidently passes through M $(a\cos\theta, b\sin\theta)$. Tangent at M to parabola is

$$x\left(\frac{\cos^3\theta}{a\times a\cos\theta}\right)^{\frac{1}{3}} + y\left(\frac{\sin^3\theta}{b\times b\sin\theta}\right)^{\frac{1}{3}} = 1,$$

or

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1,$$

which is the tangent to the ellipse at M; therefore the parabola touches the ellipse at M.

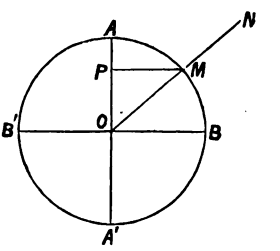
11261. (Professor KEELHOFF.)—Etant donnée une circonférence O, on prolonge un rayon quelconque OM de la longueur $MN = n\,.\,MP$, où MP est la perpendiculaire abaissée de l'extrémité M du rayon sur un diamètre fixe AA'. (1) Démontrer que le lieu du point N est un limaçon de PASCAL; (2) trouver l'aire de la courbe; et (3) trouver la longueur de la courbe, dans le cas de $n = 1$.

Solution by Prof. ZERR, M.A.; W. J. GREENSTREET, M.A.; *and others.*

Let $ON = r$, $OM = a$, $\angle MOB = \theta$,

$$MN = n\,.\,MP = an\cos\theta,$$

$$ON = r = a + an\cos\theta = a(n\cos\theta + 1).$$



(1) From Arts. 136, 163, pp. 192, 237 of WILLIAMSON'S *Int. Calc.*, we have

(1) A = area of limaçon, $r = a\cos\theta + b$,

$$A = \pi\left(\tfrac{1}{2}a^2 + b^2\right);$$

(2) $$S = \int\left\{(a+b)^2\cos^2\tfrac{1}{2}\theta + (a-b)^2\sin^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}}d\theta.$$

Putting an for a, and a for b in the above, we get

$$A = \pi a^2\left(\tfrac{1}{2}n^2 + 1\right), \quad S = \int\left\{a^2(n+1)^2\cos^2\tfrac{1}{2}\theta + a^2(n-1)^2\sin^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}}d\theta,$$

when $n = 1$, $A = \frac{3}{2}\pi a^2$,

$$S = 2a\int\cos\tfrac{1}{2}\theta\,d\theta = 4a\sin\tfrac{1}{2}\theta + \text{constant} = 8a \text{ for whole length.}$$

6014. (Professor MATZ, M.A.)—There is a circular field whose diameter is $2m$ (= 100) feet; find the length of a rope fastened at a point n (= 60) feet from the centre of the field, in order that a cow, tied by the

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tail to the rope, can graze over exactly one-a^{th} $(= \frac{1}{a\pi})$ of an acre of the field.

Solution by H. J. Woodall.

Let x = radius of inner circle = length of rope,

$OP = n$, $OQ = m$, $PQ = x$, $QPA = \theta$,

A = area of AQBQ′

$= x^2(\pi - \theta) + QOA \cdot m^2 - 2\Delta QOP$,

$\cos\theta = (m^2 - n^2 - x^2)/2nx$;

$QOA = \arcsin(2\Delta/mn)$;

where Δ = area QOP;

therefore

$$A = x^2 \operatorname{arc\,cos}\left\{(n^2 + x^2 - m^2)/2nx\right\} + m^2 \operatorname{arc\,sin}(2\Delta/mn) - 2\Delta.$$

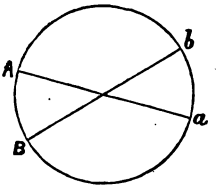
Solve for x, where

$$\Delta = \left\{s(s-m)(s-n)(s-x)\right\}^{\frac{1}{2}} \text{ and } s = \tfrac{1}{2}(m+n+x).$$

11495. (Professor Aiyar.)—Three points are taken at random on the circumference of a circle: prove that the probability of their lying on the same semicircle is $\frac{3}{4}$.

Solution by H. W. Curjel, B.A.; Professor Zerr, M.A.; *and others.*

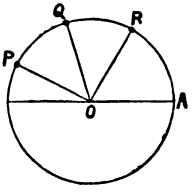
Let A, B be any position of two of the points. Draw the diameters Aa, Bb; then the chance that C is not on the same side of Aa as B is $\frac{1}{2}$; also the chance that C is not on the same side of Bb as A is $\frac{1}{2}$; hence the chance that C is in arc ab is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; and the chance that A, B, C are not on the same semicircle is $\frac{1}{4}$; therefore the chance that A, B, C lie on the same semicircle is $1 - \frac{1}{4} = \frac{3}{4}$.



[Prof. Zerr's solution is as follows:—

Let $OA = a$, $\angle POA = \theta$, $\angle QOA = \psi$,

$\angle POA = \phi$;



then the limits of θ are 0 and 2π, of ϕ, for semi-circumference $\theta - \pi$ and θ, for circumference 0 and θ; of ψ, ϕ and θ; therefore

$$p = \int_0^{2\pi}\int_{\theta-\pi}^{\theta}\int_{\phi}^{\theta} a\,d\theta\,a\,d\phi\,a\,d\psi \Big/ \int_0^{2\pi}\int_0^{\theta}\int_{\phi}^{\theta} a\,d\theta\,a\,d\phi\,a\,d\psi$$

$$= \tfrac{3}{4}.]$$

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5235. (Rev. W. A. WHITWORTH, M.A.)—Prove that the total number of signals which can be made with n different flags on s different masts is one less than the coefficient of x^n in the expansion of

$$n!\, e^x (1-x)^{-s}.$$

Solution by W. J. GREENSTREET, M.A.

If we use x different flags, we can select them in ${}_nC_x$ ways, and distribute them in $s(s+1)(s+2)\ldots(s+x-1)$ ways; therefore total number

$$= n!\left\{\frac{1}{n!}+\frac{s}{1!\,(n-1)!}+\frac{s(s+1)}{2!\,(n-2)!}+\frac{s(s+1)(s+2)}{3!\,(n-3)!}+\ldots\right\} \text{ to } (n+1) \text{ terms}$$

$$= \text{coefficient of } x^n \text{ in } n!\left\{1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right\}\left\{1+\frac{s}{1!}x+\frac{s(s+1)}{2!}x^2+\ldots\right\}$$

$$= \text{coefficient of } x^n \text{ in } n!\, e^x (1-x)^{-s};$$

therefore, excluding the case of no signal on either mast,

the total number = coefficient of x^n in $n!\, e^x (1-x)^{-s}$ less by one.

5776. (R. A. ROBERTS, M.A.)—Find the locus of the centroid of an equilateral triangle (1) inscribed in a conic, (2) circumscribed about a conic, (3) self-conjugate with regard to a conic.

Solution by W. J. GREENSTREET, M.A.

(1) The centroid and circum-centre coincide; therefore, equating coordinates, we get, if $\Sigma . \alpha = 0$,

$$x = \frac{a(a^2-b^2)}{a^2+3b^2}\cos\delta; \quad y = \frac{-b(a^2-b^2)}{b^2+3a^2}\sin\delta;$$

therefore locus is easily seen to be

$$\frac{x^2}{a^2}(a^2+3b^2)^2+\frac{y^2}{b^2}(b^2+3a^2)^2 = (a^2-b^2)^2.$$

(2) From the invariant relations between the conic and the circum- and polar-circles, we have, x_1y_1 being the centre of the circum-circle,

$$R^4-2R^2(x^2+y^2+a^2+b^2)+(x^2+y^2)^2+c^4 = 2c^2(x^2-y^2),$$

$x^2+y^2 = a^2+b^2+$ (the rectangle under the segments of the perpendiculars) (SALMON, p. 341)

$= a^2+b^2-\frac{1}{2}R^2$ in an equilateral triangle.

Therefore, eliminating ρ,

$$(3x^2+3y^2-a^2-b^2)^2 = 4(a^2x^2+b^2y^2+a^2b^2).$$

(3) R, r being circum- and in-radii, the invariants give us

$$x^2+y^2 = a^2+b^2+R^2, \quad \text{(SALMON, p. 342)},$$

$$b^2x^2+a^2y^2-a^2b^2 = (a^2+b^2)r^2 = \tfrac{1}{4}(a^2+b^2)R^2;$$

therefore locus is $(a^2-3b^2)x^2-(3a^2-b^2)y^2 = (a^2-b^2)^2$.

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11266. (Professor CAVALLIN.)—A circle through the centre of a rectangular hyperbola cuts the curve in the points A, B, C, D. Prove that the circle circumscribing the triangle formed by the tangents at A, B, C passes through the centre of the hyperbola, and has its centre at the point on the hyperbola diametrically opposite to D.

Solution by W. J. GREENSTREET, M.A.

Let $xy = a^2$ be the equation of the curve, referred to the asymptotes as axes, and let t_1, t_2, t_3, t_4 be the parameters of A, B, C, D respectively, O the centre. Then AB, CD are $y - mx - n = 0$, $y - m'x - n' = 0$; and

$$(y - mx - n)(y - m'x - n') + \lambda(xy - a^2) = 0$$

is any circle through O; whence we get

$$nn' = \lambda a^2, \quad mm' = 1, \quad \lambda = m + m'.$$

The abscissæ of A, B are the roots of $mx^2 + nx - a^2 = 0$; therefore

$$t_1 t_2 = -1/m, \quad t_1 + t_2 = -n/ma;$$

similarly $$t_3 t_4 = -1/m', \quad t_3 + t_4 = -n'/m'a;$$

therefore $$t_1 t_2 t_3 t_4 = 1,$$

and $$t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4 + t_2 t_1 = (t_1 + t_2)(t_3 + t_4) + t_1 t_2 + t_3 t_4 = 0 \ldots (1).$$

The tangents at A and B meeting at C′ are

$$x/t_1 + t_1 y = 2a, \quad x/t_2 y + t_2 y = 2m;$$

therefore C′ is $$\left(2a\frac{t_1 t_2}{t_1 + t_2}, \frac{2a}{t_1 + t_2}\right);$$

and D′, diametrically opposite D, is $-at_4, -a/t_4$; K, the foot of the perpendicular from D′ on OC′, is found to be

$$-a(t_4^{-1} + t_1 t_2 t_4)/(t_1 t_2 + t_1^{-1} t_2^{-1}).$$

Eliminating t_3 from (1),

$$(t_1 + t_2)\left(t_4 + \frac{1}{t_1 t_2 t_4}\right) = -\left(t_2 t_1 + \frac{1}{t_1 t_2}\right),$$

$$\left(\frac{1}{t_4} + t_1 t_2 t_4\right)(t_1 + t_2) = -t_1 t_2\left(t_1 t_2 + \frac{1}{t_1 t_2}\right);$$

whence $$\frac{t_1 t_2}{t_1 + t_2} = -\left(\frac{1}{t_4} + t_1 t_2 t_4\right) \Big/ \left(t_1 t_2 + \frac{1}{t_1 t_2}\right),$$

or half the abscissa of C′ is equal to the abscissa of K; *i.e.*, K is mid-point of C′; therefore, &c.

5106. (J. F. MOULTON, M.A.)—A ray of light is refracted through a prism in a principal plane. Show that, if the dispersion of two neighbouring colours be a minimum,

$$\frac{\sin(3\phi' - 2i)}{\sin\phi'} = 1 - \frac{2}{\mu^2}.$$

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Solution by Professor ZERR.

From PARKINSON'S *Optics*, Art. 169, p. 163, we have, for a minimum,

$$\tan^2\psi = \tan\phi' \tan\psi' \quad \ldots\ldots(1),$$

$$\sin\psi = \mu \sin\psi', \quad \phi' + \psi' = i \quad \ldots\ldots(2, 3).$$

From (1), we get $\sin^2\psi = \sin\phi' \sin\psi' / \cos(\phi' - \psi')$.

This, when substituted in (2), gives

$\mu^2 \sin\psi' \cos(\phi' - \psi') = \sin\phi'$, or $\mu^2 \sin(i - \phi') \cos(2\phi' - i) = \sin\phi'$, from (3).

Multiplying the last equation by 2, subtracting each member from $\mu^2 \cos(i - \phi') \sin(2\phi' - i)$, and transposing, we obtain

$$\mu^2 \{\sin(2\phi' - i) \cos(i - \phi') - \cos(2\phi' - i) \sin(i - \phi')\}$$
$$= \mu^2 \{\sin(2\phi' - i) \cos(i - \phi') + \cos(2\phi' - i) \sin(i - \phi')\} - 2\sin\phi',$$

or
$$\mu^2 \sin(3\phi' - 2i) = \mu^2 \sin\phi' - 2\sin\phi',$$

or
$$\frac{\sin(3\phi' - 2i)}{\sin\phi'} = 1 - \frac{2}{\mu^2}.$$

11162. (Professor ZERR, M.A.)—Prove that the radius of curvature at any point on the ellipse or hyperbola is equal to the product of the focal distance divided by the perpendicular from the centre on the tangent to the point.

Solution by H. J. WOODALL; R. KNOWLES, B.A.; *and others.*

By CARR'S *Synopsis* (1259), $\rho = CD^2/PF$, and, by (1213), $CD^2 = PS.PS'$;

and PF = perpendicular from P on diameter conj. to CP
= perpendicular from C on tangent at P.

8947. (The late W. J. C. SHARP, M.A.)—Prove that the lines of curvature on a developable surface are either tangents or evolutes of the cuspidal edge.

Solution by H. J. WOODALL.

The developable being the surface generated by the tangent lines of the cuspidal edge, and since a tangent plane to a developable touches that developable along a certain line, it follows that normals to the developable at points on a generating line meet (at infinity); consequently generating lines, *i.e.*, tangent lines of the cuspidal edge, are lines of curvature of the developable.

The other lines of curvature cut these tangent lines at right angles, *i.e.*, are the orthogonal trajectory of these tangent lines, and hence are the evolutes of the cuspidal edge.

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9134. (Prof. MAHENDRA NATH RAY, M.A., LL.B.)—Show that the radius of gyration of a circular arc AB of uniform density about a line through its middle point at right angles to its plane, is represented by the chord of half the arc cut off by the line drawn through the centre of gravity of the arc parallel to the chord AB.

Solution by H. J. WOODALL.

$$ML = m \cdot 2r^2(1-\sin\tfrac{1}{2}\theta/\tfrac{1}{2}\theta) = m\rho^2;$$

therefore $\rho = r\,2^{\frac{1}{2}}(1-\sin\frac{1}{2}\theta/\frac{1}{2}\theta)^{\frac{1}{2}}$ (CARR, *Synopsis*, 6027).

Mid-point of arc is C, centre of gravity is G; therefore

$$CG = r(1-\sin\tfrac{1}{2}\theta/\tfrac{1}{2}\theta);$$

therefore length of chord of arc mentioned $\left\{2r^2(1-\sin\frac{1}{2}\theta/\frac{1}{2}\theta)\right\}^{\frac{1}{2}} = \rho$.

11381. (Professor MICHEL.)—Soit AC une corde mobile dans une circonférence Δ, de rayon R. Sur AB on prend un point I tel que

$$IA^2 + IB^2 + K \cdot AB^2 = 2(1+2K)R^2;$$

K désignant une constante donnée. Trouver le lieu décrit par ce point I.

Solution by Professors MOREL, BEYENS, *and others*.

Prenons I à l'intérieur de Δ. Soit CC′ la symétrique de B par rapport à OI. $\alpha = \angle AOC - \angle AIC$.

On a $(AC)^2 = 2R^2(1-\cos\alpha) = (AI)^2 + (IB)^2 - 2AI \cdot IB\cos\alpha$,

d'où $(AI)^2 + (IB)^2 - 2R^2 = 2\cos\alpha\,(AI \cdot IB - R^2)$.

Or $AB = AI + IB$.

La relation de l'énoncé donne

$$(K+1)\left\{(AI)^2 + (IB)^2 - 2R^2\right\} = -2K\,(AI \cdot IB - R^2).$$

Donc $\cos\alpha = \text{constante}$.

Or $x = 2OIM$, &c.

8973. (The late W. J. C. SHARP.)—If

$$(n)^{(r)} \text{ denote } n(n-1)\ldots\ldots(n-r+1),$$

show that $n^r = (n)^{(r)} + {}^rA_1\,(n)^{(r-1)} + {}^rA_2\,(n)^{(r-2)} + \&c.,$

where $${}^rA_p = \frac{(r-p)^{r-1}}{(r-p-1)!} - \frac{(r-p-1)^{r-1}}{1!\,(r-p-2)!} + \frac{(r-p-2)^{r-1}}{2!\,(r-p-3)!} - \frac{(r-p-3)^{r-1}}{3!\,(r-p-4)!}$$
$$= \Delta^{r-p}0^r \div (r-p)! \qquad + \&c.$$

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Solution by H. J. WOODALL.

$$\phi(n) = \phi(0) + \Delta\phi(0)(n) + \frac{\Delta^2\phi(0)}{1.2}(n)^{(2)} + \ldots$$

$$= (n)^r + \frac{\Delta^{r-1}\phi(0)}{(r-1)!}(n)^{(r-1)} + \&c. \quad \text{(by reversing)},$$

$$\phi(n) = n^r, \quad \phi(0) = 0^r;$$

therefore $$n^r = (n)^{(r)} + \frac{\Delta^{r-1}0^r}{(r-1)!}(n)^{(r-1)} + \frac{\Delta^{r-2}0^r}{2!\,(r-2)!}(n)^{(r-2)} + \ldots;$$

hence it remains to evaluate $^rA_p = \frac{\Delta^{r-p}0^r}{(r-p)!}$.

$$\Delta^n x^m = (x+n)^m - n(x+n-1)^m + {}^nC_2(x+n-2)^m - \&c.;$$

$$\therefore \quad \Delta^{r-p}0^r = (r-p)^r - (r-p)(r-p-1)^r + {}_{r-p}C_2(r-p-2)^r - \&c.;$$

$$\therefore \quad \frac{\Delta^{r-p}0^r}{r-p!} = \frac{(r-p)^{r-1}}{(r-p-1)!} - \frac{(r-p-1)^{r-1}}{1!\,(r-p-2)!} + \frac{(r-p-2)^{r-1}}{2!\,(r-p-3)!} - \&c.$$

11279. (Rev. T. ROACH, M.A.)—Prove that, in an equation of the n^{th} degree, expressed in its simplest form, if p be the numerical value of the greatest negative coefficient, and x^{n-r} the highest power of x which has a negative coefficient, then $(1+p)^{1/r}$ is a superior limit of the positive roots.

Solution by the PROPOSER.

Let $f(x) = 0$ be the equation, then, since x^{n-r} is the first negative term, $f(x)$ will be positive for a positive value of x, if

$$x^n + p_1x^{n-1} + \ldots - p(x^{n-r} + x^{n-r-1} + \ldots + x^2 + x + 1) \text{ be positive};$$

a fortiori if

$$x^n + x^{n-1} + x^{n-2} + \ldots - p(x^{n-r} + x^{n-r-1} + \ldots + x^2 + x + 1) \text{ be positive};$$

therefore if $$\frac{x^{n-r+1}(x^r-1)}{x-1} - p\frac{(x^{n-r+1}-1)}{x-1} \text{ be positive};$$

therefore, x being greater than 1, if

$$x^{n-r+1}(x^r-1) - p(x^{n-r+1}-1) \text{ be positive};$$

a fortiori if $$x^{n-r+1}(x^r-1-p) \text{ be positive};$$

therefore if $$x^r - 1 - p \text{ be positive};$$

therefore if $$x > (1+p)^{1/r};$$

therefore $(1+p)^{1/r}$ is a superior limit of the positive roots of $f(x) = 0$.

The usual limit given is $1 + (p)^{1/r}$ and $1 + (p)^{1/r} > (1+p)^{1/r}$

if $$(1 + p^{1/r})^r > 1 + p;$$

if $1 + p + \ldots > 1 + p$, therefore, the above limit is closer.

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7381. (A. MARTIN, M.A., LL.D.)—If $F(x)$ denote any function of x, prove that, when n is greater than any assignable quantity,

$$\int[F(x)]\,dx = x/n\,[F(x/n) + F(2x/n) + F(3x/n) + \ldots + F(nx/n)].$$

Solution by H. J. WOODALL.

We have, from summation theory of integration,

$$\int_a^b [F(x)]\,dx = h\,[F(a) + F(a+h) + F(a+2h) + \ldots + F(b-h)],$$

where h is small, and n (= number of terms + 1) is large, and $nh = b-a$. Now put $h = x/n = a$; then the above becomes

$$= x/n\,[F(x/n) + F(2x/n) + \ldots + F(nx/n)],$$

where we add a term $x/n\,F(nx/n)$ on (which will not alter value of series, provided x and $F(x)$ are not infinite).

11347. (R. LACHLAN, M.A.)—Show that the number of ways in which n boys may be arranged, in (1) a row, (2) a circle, so that r particular boys may be separated, are

$$\frac{(n-r)!\,(n-r+1)!}{(n-2r+1)!}, \quad \frac{(n-r)!\,(n-r-1)!}{(n-2r)!}.$$

Solution by H. J. WOODALL; Prof. NILKANTHA SARKAR; *and others.*

(1) This is done by placing $n-r$ boys, and then choosing r out of the $n-r+1$ places for the remaining r boys,

$$\text{No. of ways} = (n-r)! \times (n-r+1)!/(n-2r+1)!$$

(2) There are $n-r$ places for the remaining r boys,

$$\text{No. of ways} = (n-r-1)! \times (n-r)!/(n-2r)!$$

11348. (MORGAN BRIERLEY.)—Find x, y, z, from

$$(x+y)\,z = a\,(=56, \text{ say}), \quad (x+z)\,y = b\,(=50), \quad (x+y)\,x = c\,(=36).$$

Solution by R. KNOWLES, B.A.; W. GREENSTREET, M.A.; *and others.*

Adding and subtracting (1) and (2), we obtain $xz = s-b$, $xy = s-a$;

$$\therefore \quad x = (s-b)/c + z = (s-a)/y; \qquad \therefore \quad y = (s-a)\,z/(s-b);$$

and substituting in (1), we get

$$x = \pm\left[(s-a)\,(s-b)/(s-c)\right]^{\frac{1}{2}}, \quad y = \pm\left[(s-a)\,(s-c)/(s-b)\right]^{\frac{1}{2}},$$

$$z = \pm\left[(s-b)\,(s-c)/(s-a)\right]^{\frac{1}{2}};$$

and this, with the specified numbers, gives $x = 3$, $y = 5$, $z = 7$.

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11273. (A. E. DANIELS, B.A.)—Prove that the area of the triangle formed by the three normals to an ellipse at the points whose eccentric angles are α, β, γ, is given by

$$\frac{(a^2-b^2)^2}{4ab}\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha)\tan\tfrac{1}{2}(\alpha-\beta) \times\left[\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\right]^2.$$

Solution by Professor ZERR.

The equations to the normals are

$$\frac{ax}{\cos\alpha}-\frac{by}{\sin\alpha}=a^2-b^2,\quad \frac{ax}{\cos\beta}-\frac{by}{\sin\beta}=a^2-b^2\ldots\ldots\ldots\ldots(1,2),$$

$$\frac{ax}{\cos\gamma}-\frac{by}{\sin\gamma}=a^2-b^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The points of intersection of (1), (2); (1), (3); (2), (3) are

$$\left\{\frac{a^2-b^2}{a}\left(\frac{\cos\alpha\cos\beta(\sin\alpha-\sin\beta)}{\sin(\alpha-\beta)}\right),\ \frac{a^2-b^2}{b}\left(\frac{\sin\alpha\sin\beta(\cos\beta-\cos\alpha)}{\sin(\beta-\alpha)}\right)\right\},$$

$$\left\{\frac{a^2-b^2}{a}\left(\frac{\cos\alpha\cos\gamma(\sin\alpha-\sin\gamma)}{\sin(\alpha-\gamma)}\right),\ \frac{a^2-b^2}{b}\left(\frac{\sin\alpha\sin\gamma(\cos\gamma-\cos\alpha)}{\sin(\gamma-\alpha)}\right)\right\},$$

$$\left\{\frac{a^2-b^2}{a}\left(\frac{\cos\beta\cos\gamma(\sin\beta-\sin\gamma)}{\sin(\beta-\gamma)}\right),\ \frac{a^2-b^2}{b}\left(\frac{\sin\beta\sin\gamma(\cos\gamma-\cos\beta)}{\sin(\gamma-\beta)}\right)\right\}.$$

Substituting these values in the expression for the area of the triangle

$$\tfrac{1}{2}\left\{y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)\right\},$$

and reducing, the required result follows.

5652. (J. HAMMOND, M.A.)—If

$$\phi(x)=A_0+A_1x+A_2x^2+\ldots \textit{ ad inf.},$$

sum the infinite series

$$A_0+\tfrac{1}{2}(A_0+A_1)+\tfrac{1}{3}(A_0+A_1+A_2)+\tfrac{1}{4}(A_0+A_1+A_2+A_3)+\&c.,$$

$$A_0-\tfrac{1}{2}(A_0-A_1)+\tfrac{1}{3}(A_0-A_1+A_2)-\tfrac{1}{4}(A_0-A_1+A_2-A_3)+\&c.$$

Supposing the series for $\phi(x)$ to be convergent, find the condition that the first of these derived series shall be convergent.

Solution by H. J. WOODALL.

$$S=A_0+\tfrac{1}{2}(A_0+A_1)+\tfrac{1}{3}(A_0+A_1+A_2)+\tfrac{1}{4}(A_0+A_1+A_2+A_3)+\&c.,$$

$$S(x)=A_0x+\tfrac{1}{2}(A_0+A_1)x^2+\tfrac{1}{3}(A_0+A_1+A_2)x^3+\&c.,$$

$$\frac{dS}{dx}=A_0+(A_0+A_1)x+(A_0+A_1+A_2)x^2+\&c.=\phi(x)+x\frac{dS}{dx};$$

therefore

$$S=\int_0^1\frac{\phi(x)}{1-x}dx,$$

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This series

$$\Sigma = A_0 - \tfrac{1}{2}(A_0 - A_1) + \tfrac{1}{3}(A_0 - A_1 + A_2) - \tfrac{1}{4}(A_0 - A_1 + A_2 - A_3) + \&c.,$$

when similarly treated, gives $\Sigma = \int_0^1 \frac{\phi(x)}{1+x}\,dx$.

Condition for convergency.

Compare with series for $-\log(1-x)$, which is convergent if $x < 1$.

S_∞ will be convergent if $S_n < 1$, where $S_n = A_1 + A_2 + \ldots + A_n$.

Hence $\phi(1) = S_\infty$ must be less than 1.

2040. (The late Professor WOLSTENHOLME, Sc.D.)—The perimeter $(2a)$ of a triangle is given, and all values of the sides for which the triangle is real are equally probable; prove that the mean value of the circum-radius is 5 times, and that of the ex-radius 7 times, the mean value of the in-radius.

Solution by Professor ZERR.

Let x, $y-x$, $2a-y$, be the sides: then, for (1) the circum-circle, (2) the ex-circle, and (3) the in-circle, we have

$$\Delta_1 = \frac{1}{4\sqrt{a}} \int_a^{2a}\int_{y-a}^{a} \frac{x(y-x)(2a-y)}{\{(a-x)(y-a)(a-y+x)\}^{\frac{1}{2}}}\,dy\,dx \Big/ \int_a^{2a}\int_{y-a}^{a} dy\,dx$$

$$= \frac{1}{2a^2\sqrt{a}} \int_a^{2a}\int_{y-a}^{a} \frac{x(y-x)(2a-y)}{\{(a-x)(y-a)(a-y+x)\}^{\frac{1}{2}}}\,dy\,dx$$

$$= \frac{\pi}{16a^2\sqrt{a}} \int_a^{2a} \left\{ 8a(y-a)^{\frac{1}{2}} + \frac{(2a-y)^2}{(y-a)^{\frac{1}{2}}} \right\} (2a-y)\,dy = \frac{4\pi a}{21};$$

$$\Delta_2 = \sqrt{a} \int_a^{2a}\int_{y-a}^{a} \left\{ \frac{(a-x)(a-y+x)}{y-a} \right\} dy\,dx \Big/ \int_a^{2a}\int_{y-a}^{a} dy\,dx$$

$$= \frac{2}{a\sqrt{a}} \int_a^{2a}\int_{y-a}^{a} \left\{ \frac{(a-x)(a-y+x)}{y-a} \right\}^{\frac{1}{2}} dy\,dx$$

$$= \frac{\pi}{4a\sqrt{a}} \int_a^{2a} \frac{(2a-y)^2}{(y-a)^{\frac{1}{2}}}\,dy = \frac{4\pi a}{15};$$

$$\Delta_3 = \frac{1}{\sqrt{a}} \int_a^{2a}\int_{y-a}^{a} \{(a-x)(y-a)(a-y+x)\}^{\frac{1}{2}}\,dy\,dx \Big/ \int_a^{2a}\int_{y-a}^{a} dy\,dx$$

$$= \frac{2}{a^2\sqrt{a}} \int_a^{2a}\int_{y-a}^{a} \{(a-x)(y-a)(a-y+x)\}^{\frac{1}{2}}\,dy\,dx$$

$$= \frac{\pi}{4a^2\sqrt{a}} \int_a^{2a} (2a-y)^2 (y-a)^{\frac{1}{2}}\,dy = \frac{4\pi a}{105}.$$

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11089. (Professor RAMASWAMI AIYAR.)—Given a conic referred to its axes, show that constants λ, μ may be determined, such that the circle passing through any point (h, k) on the conic, and having its centre at $(\lambda h, \mu k)$, may cut the conic again at the vertices of an equilateral triangle.

Solution by the PROPOSER.

Let the circle $C \equiv (x-\lambda h)^2 + (y-\mu k)^2 - r^2 = 0$ pass through the point (h, k) on the conic $S \equiv ax^2 + by^2 - 1$; and cut the conic, and therefore also the equilateral hyperbola $2S = (a+b)\,C$ at the vertices of an equilateral triangle.

By a well-known property, therefore, the line joining $(\lambda h, \mu k)$ to (h, k) must be a diameter of the hyperbola; that is, the point $[\frac{1}{2}(\lambda+1)h, \frac{1}{2}(\mu+1)k]$ satisfies the equations, giving, for the centre of the hyperbola, $ax = (a+b)(x-\lambda h)$, $by = (a+b)(y-\mu k)$; substituting, we see that λ and μ are independent of h and k.

11181. (Professor BÈNÉZECH.)—Trouver quatre nombres entiers tels que leur somme soit égale à la somme obtenue en ajoutant le produit du plus grand par le plus petit nombre au produit des deux autres.

Solution by H. W. CURJEL, B.A.; Professor ZERR, M.A.; *and others.*

Let w, x, y, z be the four numbers, in descending order of magnitude; then
$$wz + xy = x + y + z + w;$$
therefore
$$(wz - z - w + 1) = x + y - xy - 1 + 2;$$
therefore
$$(w-1)(z-1) = 2 - (x-1)(y-1).$$
But $(w-1)(z-1)$ is not negative; therefore
$$x = y = z = w = 2,$$
or $z = 1$, $x, y = 3, 2$, $w =$ any whole number greater than 3.

2672. (R. TUCKER, M.A.)—From a point on an ellipse chords are drawn parallel to fixed straight lines; find the maximum triangle formed by the three chords of section.

Solution by H. J. WOODALL.

Two chords are
$$x/a \cos\tfrac{1}{2}(\phi_1+\phi_2) + y/b \sin\tfrac{1}{2}(\phi_1+\phi_2) = \cos\tfrac{1}{2}(\phi_1-\phi_2),$$
and
$$x/a \cos\tfrac{1}{2}(\phi_1+\phi_3) + y/b \sin\tfrac{1}{2}(\phi_1+\phi_3) = \cos\tfrac{1}{2}(\phi_1-\phi_3),$$

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rallel to given lines; therefore

$$\phi_1 + \phi_2 = \alpha \text{ and } \phi_1 + \phi_3 = \beta;$$
$$\phi_2 + \phi_3 = \alpha + \beta - 2\phi_1.$$

), (ϕ_2), (ϕ_3) is

$$= \pm \tfrac{1}{2}ab \begin{vmatrix} \cos\phi_1, & \sin\phi_1, & 1 \\ \cos\phi_2, & \sin\phi_2, & 1 \\ \cos\phi_3, & \sin\phi_3, & 1 \end{vmatrix} = \pm \tfrac{1}{2}ab \sin\tfrac{1}{2}(\alpha-\beta) \left\{\cos\tfrac{1}{2}(\alpha+\beta-\phi_1) - \cos\tfrac{1}{2}(\alpha-\beta)\right\}.$$

This is a maximum (1) if

$\cos\frac{1}{2}(\alpha+\beta-\phi_1) - \cos\frac{1}{2}(\alpha-\beta)$ is $+$, when $\cos\frac{1}{2}(\alpha+\beta-\phi_1) = 1$;

therefore $\phi_1 = \alpha + \beta$;

(2) if $\cos\frac{1}{2}(\alpha+\beta-\phi_1) - \cos\frac{1}{2}(\alpha-\beta)$ is $-$, when $\cos\frac{1}{2}(\alpha+\beta-\phi_1) = 0$;

therefore $\phi_1 = \alpha + \beta \pm \pi$.

11322. (R. Knowles, B.A.)—From a point T on the latus-rectum of a parabola, tangents are drawn to meet the parabola in P, Q; the circle TPQ cuts the curve again in CD; the diameter of the parabola through T meets the curve in R; K is the pole of CD, with respect to the circle; O is the centre of curvature at R; prove that OK is a diameter of the parabola.

Solution by Rev. T. Galliers, M.A.; Prof. Radikanshuan; *and others.*

Let the parabola be $y^2 = 4ax$, and let T be (k, a); then the equations of the circle TPQ and of the line CD will be

$$a(x^2+y^2) - (k^2+2a^2)x + a^3 = 0, \quad 2ax + ky + 2a^2 = 0 \ldots\ldots (1, 2);$$

and the polar of (X, Y) with regard to (1), is

$$\left\{aX - \tfrac{1}{2}(k^2+2a^2)\right\}x + aYy + a^3 - \tfrac{1}{2}(k^2+2a^2)X = 0 \ldots\ldots (3).$$

If (X, Y) be K, we find, by comparing (2) and (3), that $Y = -k^3/4a^2$, which is known to be the ordinate of O; hence OK is a diameter of the parabola.

11427. (R. Lachlan, M.A.)—If the points of contact of the three tangents which can be drawn from the point P to the cardioid $r = a(1+\cos\theta)$ be collinear, prove that (1) the locus of P is the circle $r + 2a\cos\theta = 0$; and (2) if the feet of the three normals which can be drawn from P be collinear, the locus of P is the circle $3r = a\cos\theta$.

Solution by R. F. Davis, M.A.; Professor Aiyar; *and others.*

(1) Any point on the cardioid $r = a(1-\cos\theta)$ may be conveniently defined by the single variable $t\,(\equiv \tan\frac{1}{2}\theta)$. To find the criterion that

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the equation $Kt^3 + Lt^2 + Mt + N = 0$ should represent three collinear points. At the points of intersection of any straight line

$$\lambda r \cos\theta + \mu r \sin\theta = 2a$$

with the curve, we have $\sin^2\frac{1}{2}\theta\,(\lambda\cos\theta + \mu\sin\theta) = 1$; dividing each side by $\cos^4\frac{1}{2}\theta$,

$$t^2\left\{\lambda(1-t^2) + 2\mu t\right\} = (1+t^2)^2,$$

or

$$t^4(1+\lambda) - 2\mu t^3 + (2-\lambda)t^2 + 1 = 0,$$

in which it is to be observed that the term involving t is absent.

Identifying this with the equation $(Kt^3 + Lt^2 + Mt + N)(Mt - N) = 0$, we get

$$1+\lambda : 2-\lambda : 1 = KM : M^2 - LN : -N^2;$$

hence $KM + M^2 - LN + 3N^2 = 0$, which is the required criterion.

(2) To find the points the tangents at which pass through (ρ, ϵ).

Since the angle between the radius-vector and tangent $= \frac{1}{2}\theta$, the equation to the tangent at any point (r, θ) is $R\sin(\frac{3}{2}\theta - \Theta) = 2a\sin^3\frac{1}{2}\theta$.

If this passes through (ρ, ϵ), we have $\rho\sin(\frac{3}{2}\theta - \epsilon) = 2a\sin^3\frac{1}{2}\theta$, or

$$(3\sin\tfrac{1}{2}\theta - 4\sin^3\tfrac{1}{2}\theta)\cos\epsilon - (4\cos^3\tfrac{1}{2}\theta - 3\cos\tfrac{1}{2}\theta)\sin\epsilon = 2a/\rho\sin^3\tfrac{1}{2}\theta,$$

$$\sin^3\tfrac{1}{2}\theta\,(4\cos\epsilon + 2a/\rho) + 4\sin\epsilon\cos^3\tfrac{1}{2}\theta = 3\cos\epsilon\sin\tfrac{1}{2}\theta + 3\sin\epsilon\cos\tfrac{1}{2}\theta.$$

Dividing by $\cos^3\frac{1}{2}\theta$, $t^3(4\cos\epsilon + 2a/\rho) + 4\sin\epsilon = 3(1+t^2)(t\cos\epsilon + \sin\epsilon)$;

$$t^3(\cos\epsilon + 2a/\rho) - 3t^2\sin\epsilon - 3t\cos\epsilon + \sin\epsilon = 0.$$

Applying the criterion, we have

$$-3\cos\epsilon(\cos\epsilon + 2a/\rho) + 9\cos^2\epsilon + 3\sin^2\epsilon + 3\sin^2\epsilon = 0,$$

$$-6a/\rho\cos\epsilon + 6 = 0, \qquad \rho = a\cos\epsilon.$$

(3) To find the points the normals at which pass through (ρ, ϵ). The equation to the normal at any point (r, θ) is

$$R\cos(\tfrac{3}{2}\theta - \Theta) = 2a\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta.$$

If this passes through (ρ, ϵ), it will be found, as before, that the equation determining t is $t^3\sin\epsilon + (2a/\rho + 3\cos\epsilon)t^2 - 3\sin\epsilon t - \cos\epsilon = 0$.

Applying the criterion, we get $3\rho + a\cos\epsilon = 0$.

[Mr. Davis believes that Mr. Lachlan has taken as the equation to the cardioid $r = a(1+\cos\theta)$ instead of $r = a(1-\cos\theta)$, and that his results should be $r + a\cos\theta = 0$, $3r = a\cos\theta$. By revolving the initial line of polar coordinates through an angle of 180°, Mr. Lachlan's results would be $r = a\cos\theta$, $3r + a\cos\theta = 0$.]

11240. (R. Knowles, B.A.)—ABCDE is a pentagon inscribed in a conic; the five points of intersection of the sides AB, BC, CD, DE, EA with the tangents at D, E, A, B, C respectively are collinear; and of course its reciprocal is true, that the five lines joining the poles of AB, BC, &c. to the points D, E, &c., are concurrent.

Solution by J. C. St. Clair; Prof. Chakrivarti; *and others.*

Since the point E is involved but once, the question asserts that a variable line (the tangent at E) always passes through a fixed point

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determined by connecting four given points A, B, C, D; which is impossible unless the pentagon be a regular one. The conic must then be a circle, and it is obvious that the tangents meet the opposite sides at infinity.

[The PROPOSER does "not agree with the above remarks that the pentagon must necessarily be a regular one, or that it is confined to the circle."]

8180. (W. J. GREENSTREET, B.A.)—If A_p is coefficient of x^p in expansion of $(a + bx + cx^2)^n$, prove that, if $m > n$, $c^m A_{n-m} = a^m A_{n+m}$.

Solution by H. J. WOODALL.

$$(a+bx+cx^2)^n = x^n\left\{1+(cx+a/x)\right\}^n,$$

$$x^n\left[1+n\,(cx+a/x)+\&c.\right];$$

then $$A_{n-m} = \frac{n!}{m!\,(n-m)!} \times a^m + \frac{n!}{(m+2)!\,(n-m+2)!}\,a^{(m+1)}c\,(m+z) + \&c.;$$

$$A_{n+m} = \frac{n!}{m!\,(n-m)!}\,c^m + \frac{n!}{(m+2)!\,(n-m+2)!}\,(m+2)\,c^{(m+1)}a + \&c.;$$

therefore $$c^m A_{n-m} = a^m A_{n+m}.$$

7721. (BELLE EASTON.) — One card out of a pack has been lost. From the remainder of the pack, thirteen cards are drawn at random, and are found to consist of two spades, three clubs, four hearts, and four diamonds. What are the respective chances that the missing card is a spade, a club, a heart, or a diamond?

Solution by H. J. WOODALL.

If missing card is a spade, probability of arrangement

$= 12.11.13.12.11.13.12.11.10.13.12.11.10/D = 11K;$

if missing card is a club, probability of arrangement

$= 13.12.12.11.10.13.12.11.10.13.12.11.10/D = 10K;$

if missing card is a heart, probability of arrangement

$= 13.12.13.12.11.12.11.10.9.13.12.11.10/D = 9K;$

if missing card is a diamond, probability of arrangement

$= 13.12.13.12.11.13.12.11.10.12.11.10.9/D = 9K;$

$D = 51!/38!$, $K = 13.12.13.12.11.13.12.11.10.12.11.10/D;$

therefore probability of missing card being a spade

$$= 11/(11+10+9+9) = \tfrac{11}{39};$$

therefore probability of missing card being a club, heart, diamond,

$$= \tfrac{10}{39},\ \tfrac{9}{39},\ \tfrac{9}{39},\ \text{respectively.}$$

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7858. (R. KNOWLES, B.A.)—From a point Q in the circumference of an ellipse, chords PQ, PR are drawn at right angles; (1) show that the locus of the pole of the chord QR is a right line; (2) as P moves along the ellipse, find the locus of the intersection of this line with the tangent at P; (3) show that this locus is also that of the pole of normal chords to the ellipse.

Solution by H. J. WOODALL.

Let P be (ϕ), Q (ϕ_1), R (ϕ_2); therefore PQ is

$$x/a \,.\, \cos\tfrac{1}{2}(\phi+\phi_1) + y/b \,.\, \sin\tfrac{1}{2}(\phi+\phi_1) = \cos\tfrac{1}{2}(\phi-\phi_1) \ldots\ldots (1);$$

PR is $x/a \,.\, \cos\tfrac{1}{2}(\phi+\phi_2) + y/b \,.\, \sin\tfrac{1}{2}(\phi+\phi_2) = \cos\tfrac{1}{2}(\phi-\phi_2) \ldots\ldots (2)$;

condition for perpendicularity of PQ and PR becomes

$$(a^2-b^2)\cos\phi\cos\tfrac{1}{2}(\phi_1+\phi_2)/(a^2+b^2) - (a^2-b^2)\sin\phi\sin\tfrac{1}{2}(\phi_1+\phi_2)/(a^2+b^2) = \cos\tfrac{1}{2}(\phi_1-\phi_2) \ldots\ldots (3);$$

and QR is

$$x/a \,.\, \cos\tfrac{1}{2}(\phi_1+\phi_2) + y/b \,.\, \sin\tfrac{1}{2}(\phi_1+\phi_2) = \cos\tfrac{1}{2}(\phi_1-\phi_2) \ldots\ldots (4);$$

Pole of QR is

$$x = a\cos\tfrac{1}{2}(\phi_1+\phi_2)/\cos\tfrac{1}{2}(\phi_1-\phi_2), \quad y = b\sin\tfrac{1}{2}(\phi_1+\phi_2)/\cos\tfrac{1}{2}(\phi_1-\phi_2);$$

substitute in (3), therefore

$$(a^2-b^2)\left[x/a \,.\, \cos\phi - y/b \,.\, \sin\phi\right] = (a^2+b^2), \text{ a straight line} \ldots\ldots (5);$$

tangent at P is $x/a \,.\, \cos\phi + y/b \,.\, \sin\phi = 1 \ldots\ldots (6)$;

(5) and (6) meet at

$$x/a \,.\, \cos\phi = a^2/(a^2-b^2), \quad y/b \,.\, \sin\phi = -b^2/(a^2-b^2).$$

Eliminating ϕ, we get $(a^2-b^2)^2 = a^6/x^2 + b^6/y^2 \ldots\ldots (7)$.

(3) Normal chord at P is one of the chords subtending a right angle at P, and therefore its pole is on (5). This pole also lies on the tangent at P whence (7) is locus of poles of normal chords.

11405. (Professor CROFTON, F.R.S.)—Prove that

$$\frac{x + 2^r\frac{x^2}{1.2} + 3^r\frac{x^3}{1.2.3} + \&c.}{1 + x + \frac{x^2}{1.2} + \&c.} = x + \frac{\Delta^2 0^r}{1.2}x^2 + \frac{\Delta^3 0^r}{1.2.3}x^3 + \ldots x^r.$$

Solution by H. J. WOODALL.

We have $u_{n+m} = u_n + m\Delta u_n + \dfrac{m(m-1)}{1.2}\Delta^2 u_n + \&c.$

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For $x = 1$, then x becomes a, and then $u_x = u'$.

$$\therefore \quad u' = u + n\Delta u + \frac{n(n-1)}{1 \cdot 2}\Delta^2 u + \quad + \frac{n!}{r!\,(n-r)!}\Delta^r u.$$

Substituting in the given fraction, and dividing out, we get the right-hand side. All the series are convergent.

11287. (E. Knowles, B.A.)—Prove that the sum of the series

$$\frac{1}{4} - \frac{2^2}{7} + \frac{2^4}{10} - \frac{2^6}{13} + \ldots \text{ ad infin.} \equiv \frac{1}{16}\left(2 - \tfrac{1}{2}\log 3^3 - \tfrac{1}{6}\pi.3^{\frac{1}{2}}\right).$$

Solution by H. W. Curjel, B.A.; Prof. Mukhopadhyay; *and others.*

Let $\frac{1}{4} - \frac{2^2}{7} + \frac{2^4}{10} - \ldots = S$, and let ω_1, ω_2 be the two imaginary cube roots of unity.

$$3(2 - 16S) = 3\left\{2 - \frac{2^4}{4} + \frac{2^7}{7} - \frac{2^{10}}{10} + \frac{2^{13}}{13} \ldots\right\}$$

$$= 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \ldots$$

$$+ \frac{1}{\omega_1}\left\{2\omega_1 - \frac{(2\omega_1)^2}{2} + \frac{(2\omega_1)^3}{3} - \frac{(2\omega_1)^4}{4} + \ldots\right\}$$

$$+ \frac{1}{\omega_2}\left\{2\omega_2 - \frac{(2\omega_2)^2}{2} + \frac{(2\omega_2)^3}{3} - \frac{(2\omega_2)^4}{4} + \ldots\right\}$$

$$= \log 3 + \frac{\log(2\omega_1 + 1)}{\omega_1} + \frac{\log(2\omega_2 + 1)}{\omega_2}$$

$$= \log 3 + \omega_2 \log(2\omega_1 + 1) + \omega_1 \log(2\omega_2 + 1)$$

$$= \log 3 - \tfrac{1}{2}\log 3 - i\sqrt{3}\log i$$

$$= \tfrac{1}{2}\log 3 - \sqrt{3}\,i2\left\{\frac{i-1}{i+1} + \tfrac{1}{3}\left(\frac{i-1}{i+1}\right)^3 + \tfrac{1}{5}\left(\frac{i-1}{i+1}\right)^5 + \ldots\right\}$$

$$= \tfrac{1}{2}\log 3 + 2\sqrt{3}\left\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots\right\}$$

$$16S = 2 - \tfrac{1}{2}\log 3^3 - \tfrac{1}{6}\pi.3^{\frac{1}{2}}.$$

5236. (R. A. Roberts, M.A.)—Prove that the circumcircle of a triangle inscribed in a conic and circumscribed to a circle touches two fixed circles having double contact with the conic.

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Solution by Professors RAMASWAMI AIYAR, SARKAR, *and others.*

Let a triangle Δ be inscribed in a conic U and circumscribed to a cirlce C; then, by a Theorem of ROBERTS, Δ is always self-conjugate with respect to a conic S; therefore the circumcircle of Δ cuts the director-circle of S orthogonally; moreover, the circumcircle of Δ, being connected with C by a relation of the form $R^2 - 2Rr = d^2$, is to be regarded as cutting the point-circle at the centre of C at a constant angle. Hence the circumcircle of Δ cuts all circles of the coaxal system, one of which is the director-circle of S, and one of whose limiting is the centre of C, at constant angles; and, of course, touches two of them. To prove that these two circles have double contact with U, let the equations of S and C be

$$S \equiv x^2/a^2 + y^2/b^2 - 1 = 0, \quad C \equiv (x-x')^2 + (y-y')^2 - r^2 = 0;$$

then

$$r^2 = (x'^2/a^2 + y'^2/b^2 - 1) \div (1/a^2 + 1/b^2),$$

since triangles self-conjugate to S can be circumscribed to C, or $\Theta = 0$. The equation of U, which is the reciprocal of C with respect to S, is easily found to be

$$U \equiv r^2(x^2/a^4 + y^2/b^4) - (xx'/a^2 + yy'/b^2 - 1)^2 = 0.$$

Now, the polar of $(x'y')$ with respect to U is

$$r^2(xx'/a^4 + yy'/b^4) - (x'^2/a^2 + y'^2/b^2 - 1)(xx'/a^2 + yy'/b^2 - 1) = 0;$$

or, reducing,

$$xx' + yy' - a^2 - b^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Thus the polar of $(x'y')$ with respect to U is the same as the polar of $(x'y')$ with respect to any circle of the coaxal system

$$\lambda(x^2 + y^2 - a^2 - l^2) + (x-x')^2 + (y-y')^2 = 0.$$

Hence any circle of this system cuts U in four points, such that two of the lines joining them intersect at $(x'y')$ the centre of C. Conversely, if any two chords of U be drawn through $(x'y')$ equally inclined to the axes of U, the circle X through their extremities is a circle of this coaxal system, and, by what was proved before, will be cut by the circumcircle of Δ at some constant angle. Making the chords coincide (with a parallel to either axis of U through the centre of C), the circle X now comes to have double contact with U; in this case, supposing that one of the vertices A of the triangle Δ or ABC is at an extremity of the chord of contact, we see that the two sides AB, AC are equally inclined to the axes of U, and therefore the circumcircle of ABC or Δ touches at A the conic U as well as the circle X.

Hence the circumcircle of Δ, *in any position whatever*, touches X, that is, either circle having double contact with the conic U, the chord of contact being a parallel through the centre of C to either axis of U.

11312. (J. C. ST. CLAIR.)—Show that, of the three systems in involution determined by four collinear points taken in pairs, the centres of any two are conjugate points of the third.

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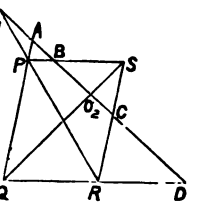
Solution by the PROPOSER.



Let A, B, C, D be the given points.

Through any pairs AC, BD draw pairs of parallel lines forming a parallelogram PQRS whose diagonals cut AD in O_1, O_2. Then

$$O_1A/O_1C = O_1P/O_1Q = O_1B/O_1D.$$

Hence $O_1A \cdot O_1D = O_1B \cdot O_1C$, and O_1 is the centre of the involution determined by AD, BC. In the same manner it is shown that O_2 is the centre of the involution AB, CD. But O_1, O_2 are conjugate points of the third system AC, BD, formed on the transversal AD by the opposite sides and diagonals of the parallelogram PQRS.

9110. (Professor ASUTOSH MUKHOPADHYAY, M.A., F.R.A.S.)—Prove that (1) the locus of the points on a system of confocal ellipses or hyperbolas, for which the eccentric angle has a constant value, is a hyperbola; (2) the envelopes of the hyperbolas, for different values of the eccentric angle, are two pairs of straight lines, passing through the foci and parallel to the bisectors of the angles between the axes, and are real when the confocals are hyperbolas, but imaginary when they are ellipses.

Solution by H. J. WOODALL.

1. Ellipse: ϕ eccentric angle; point $\{(a^2+\lambda)^{\frac{1}{2}} \cos\phi,\ (b^2+\lambda)^{\frac{1}{2}} \sin\phi\}$;

locus is $$\left(\frac{x}{\cos\phi}\right)^2 - \left(\frac{y}{\sin\phi}\right)^2 = a^2-b^2 \text{ (hyperbola).}$$

Put $\sin^2\phi = k$; therefore

$$kx^2-y^2(1-k) = (a^2-b^2)k(1-k),\quad k^2(a^2-b^2)+k(x^2+y^2-a^2+b^2)-y^2 = 0.$$

Envelope is $$4y^2(a^2-b^2)+(x^2+y^2-a^2+b^2)^2 = 0;$$

therefore $$2y(a^2-b^2)^{\frac{1}{2}} = (-1)^{\frac{1}{2}}(x^2+y^2-a^2+b^2),$$

imaginary, but passes through $\{\pm(a^2-b^2)^{\frac{1}{2}},\ 0\}$, *i.e.*, passes through the foci.

2. Hyperbola: ϕ eccentric angle; point $\{(a^2+\lambda)^{\frac{1}{2}} \sec\phi,\ (b^2+\lambda)^{\frac{1}{2}} \tan\phi\}$;

$$\left(\frac{x}{\sec\phi}\right)^2 - \left(\frac{y}{\tan\phi}\right)^2 = a^2-b^2 \text{ (hyperbola).}$$

Put $\tan^2\phi = k$; therefore $kx^2-y^2(1+k) = (a^2-b^2)k(1+k)$;

therefore $$k^2(a^2-b^2)+k(a^2-b^2-x^2+y^2)+y^2 = 0.$$

Envelope is $$4y^2(a^2-b^2) = (a^2-b^2-x^2+y^2)^2;$$

therefore $$\pm 2y(a^2-b^2)^{\frac{1}{2}} = a^2-b^2-x^2+y^2;$$

therefore $x^2 = y^2 \mp 2y(a^2-b^2)^{\frac{1}{2}} + (a^2-b^2)$, $\therefore\ x = \pm\{y \mp (a^2-b^2)^{\frac{1}{2}}\}$,

two pairs of lines, pass through $\{\pm(a^2-b^2)^{\frac{1}{2}},\ 0\}$ (foci), and are parallel to the bisectors of the angles between axes.

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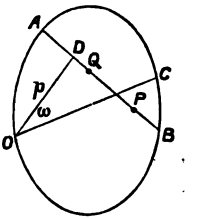
11513. (Professor CROFTON, F.R.S.)—If Σ, Σ' be the two segments into which the chord C divides the area Ω, prove that

$$\iint C^4 dp\,d\omega = 6\iint \Sigma\Sigma' dp\,d\omega.$$

Solution by Professors ZERR, RADIKANSHUAN, *and others.*



Let L = length of contour, and M = mean distance between the two points P, Q; Δ the chance that the points lie on different sides of a chord. Segment ACB = Σ, segment AOB = Σ', OD = p, $\angle$ DOC = ω, AB = C, AP = x, PQ = y. The limits of y are 0 and x, doubled; of x, 0 and C;

$$\therefore\ M = \frac{2}{\Omega^2}\iint dp\,d\omega \int_0^C \int_0^x dx y^2 dy = \frac{1}{6\Omega^2}\iint C^4 dp\,d\omega,$$

$$\Delta = 2\iint \Sigma\Sigma' dp\,d\omega \Big/ \Omega^2 \iint dp\,d\omega = \frac{2}{\Omega^2 L}\iint \Sigma\Sigma' dp\,d\omega.$$

But $\Delta = \frac{2M}{L}$; $\quad\therefore\ M = \frac{1}{\Omega^2}\iint \Sigma\Sigma' dp\,d\omega$;

$$\therefore\ \iint C^4 dp\,d\omega = 6\iint \Sigma\Sigma' dp\,d\omega.$$

[See WILLIAMSON'S *Integral Calculus*, p. 348.]

11389. (D. BIDDLE.)—Around the wheel of an Attwood's Machine, for illustrating the laws of gravitation, passes a rope, to one extremity of which is attached a weight A, and to the other extremity a pulley B, having weights a and b playing around it. Supposing

$$b = a(g+1)/(g-1), \quad \text{and} \quad B + a + b = A(g+1)/(g-1),$$

find the positions of b and B respectively at the end of each successive second, friction and the resistance of the air being disregarded.

Solution by H. W. CURJEL, B.A.

Let x be the depth of B below fixed pulley, and y the depth of b below B. Let T and T′ be the tensions of the strings. Equations of motion are

$$A/g\,\ddot{x} = T - A, \quad B/g\,\ddot{x} = B + 2T' - T,$$

$$b/g\,(\ddot{x}+\ddot{y}) = b - T', \quad a/g\,(\ddot{y}-\ddot{x}) = T' - a;$$

hence, eliminating T, T′,

$$(a+b)\,\ddot{y} + (b-a)\,\ddot{x} = g\,(b-a),$$

$$(A+B+a+b)\,\ddot{x} + (b-a)\,\ddot{y} = (B+a+b-A)\,g.$$

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Eliminating B and b,

$$g\ddot{y}+\ddot{x}=g, \quad Ag\ddot{x}+a\ddot{y}=gA;$$

therefore $$\ddot{y}=\frac{Ag(g-1)}{Ag^2-a}, \text{ and } \ddot{x}=\frac{(Ag-a)g}{Ag^2-a};$$

therefore, if the system starts from rest in position x_1, y_1,

$$x=\tfrac{1}{2}\frac{(Ag-a)g}{Ag^2-a}t^2+x_1, \quad x+y=\tfrac{1}{2}\frac{(2Ag-a-1)g}{Ag^2-a}t^2+x_1+y_1.$$

[The Proposer, though agreeing with the general method of the above solution, objects to the second equation of motion given, because in estimating the moving *mass*, B is separated from the weight depending from it, namely, 2T′. He is of opinion that the equation should be

$$(B+2T')/g\ddot{x}=B+2T'-T.$$

This, taken in conjunction with the first equation, $A/g\ddot{x}=T-A$, is in accord with the formula for obtaining the *tension*, where two falling bodies are connected by a string as in the case before us,

$$T=2MM'g/(M+M'),$$

which is readily seen to be $T=2A(B+2T')/(A+B+2T')$.

But its adoption here leads to a quadratic equation,

$$4ab\ddot{x}^2-\{(A+B)(a+b)+8ab\}g\ddot{x}+\{(B-A)(a+b)+4ab\}g^2=0,$$

whence $$\ddot{x}=\{(Ag^2+ag^2-2a)\pm[(Ag^2+ag^2-2a)^2-4a(g^2-1)(Ag-a)]^{\frac{1}{2}}\}g/\{2a(g^2-1)\}.$$

We already have $\ddot{y}=(g-\ddot{x})/g$, and $x=\frac{1}{2}\ddot{x}.t^2+x_1$,

and $$x+y=\tfrac{1}{2}(\ddot{x}+\ddot{y})t^2+x_1+y_1.$$

The conditions are fulfilled if $a=g-1$, $b=g+1$, $B=g+3$, $A=3(g-1)$.

Let $g=32{\cdot}2$, then $\ddot{x}=\cdot7472$, $\ddot{y}=\cdot9768$ of a foot.]

11527. (Professor Thiry.)—Construire le triangle rectangle dont on connaît la hauteur relative à l'hypoténuse, ainsi que la distance du sommet de l'angle droit au pied de la bissectrice d'un angle aigu.

Solution by W. J. Greenstreet, M.A.; Professor Aiyar; *and others.*

The internal and external bisectors BE, BL meet the perpendicular AF in points E and G, harmonic conjugates to A and F, and

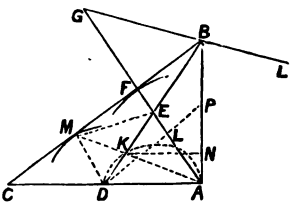
$$\angle ADE = C+\tfrac{1}{2}B = 90^\circ-\tfrac{1}{2}B;$$

$$\angle DEA = FEB = 90^\circ-\tfrac{1}{2}B;$$

$$\therefore \quad \angle ADE = DEA,$$

and $AE = AD = d$ the given length.

Hence, construct the range AEFG.



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The circle on EG as diameter cuts the line BF perpendicular to AF in B. The rest is obvious.

Otherwise: Taking BM = AB, we have AM bisected at right angles by BD, because B is bisected by BD.

It follows that ADME is a rhombus, and AE = AD.

Hence, construct a triangle FEM, having $F = 90°$, $ME = d$, $FE = h - d$. Complete the rhombus MEAD, and the intersections of AD, FM, DE, FM give the points B and C.

Or again: With centres A and D describe two circles, radii h and d respectively. Then, if $d < h < 2d$, the common tangent is the hypotenuse of the triangle required. If $d > h$, we still can draw a common tangent, but BD is the external bisector of B.

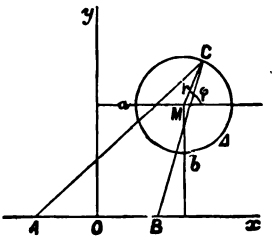
11294. (Professor Sollertinsky.)—Si le sommet C d'un triangle ABC décrit une circonférence Δ et la base AB soit fixe, l'aire ABC varie proportionnellement à la puissance de C par rapport à un cercle fixe Δ'. Dans quel cas ce cercle Δ' peut-il être remplacé par un point fixe?

Solution by Professors Schoute, Morel, *and others.*

Soit AB l'axe des x et le milieu O de AB l'origine des coordonnées rectangulaires. Représentons les cercles Δ, Δ' par

$$\left.\begin{aligned} x &= a + r\cos\phi \\ y &= b + r\sin\phi \end{aligned}\right\}$$

$$(x-\alpha)^2 + (y-\beta)^2 - \rho^2 = 0.$$



Pour AO = OB = p, la condition du problème est

$$p(b + r\sin\phi) = \mu\{(a + r\cos\phi - \alpha)^2 + (b + r\sin\phi - \beta)^2 - \rho^2\}.$$

indépendamment de ϕ. Ainsi l'on trouve $\alpha = a$, $\beta^2 - b^2 = \rho^2 - r^2$.

Si Δ' se reduit à un point on a de plus $\rho = 0$. Ce point N a les coordonnées a, $(b^2 - r^2)^{\frac{1}{2}}$.

11533. (I. Arnold.)—Construct a right-angled triangle, having given the median lines from the ends of the hypotenuse to the opposite sides.

Solution by D. Biddle; H. J. Woodall; *and others.*

Let LM, LN be the given lengths of the median lines. With M, N as centres, and LM, LN as radii, describe the semicircles PRL, QSL. Make

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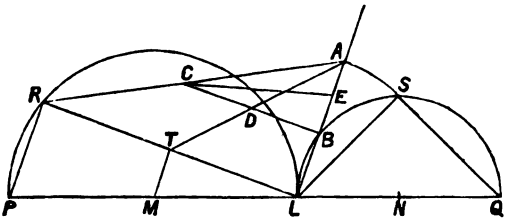
PR = LN, and QS = LM. Join LR, LS, and, drawing LA at right



angles to LR, make LA = LS. Join AR, and bisect RL in T. On A'T make AD = LN, and through D, draw BC parallel to LR. ABC is the triangle required.

For $AB^2+\frac{1}{4}BC^2 = AD^2$, and $\frac{1}{4}AB^2+BC^2 = CE^2$;

whence $\frac{15}{4}AB^2 = 4AD^2-CE^2$, and $\frac{15}{4}BC^2 = 4CE^2-AD^2$;

$\therefore$ $AB : BC = (4AD^2-CE^2)^{\frac{1}{2}} : (4CE^2-AD^2)^{\frac{1}{2}} = (4LN^2-LM^2)^{\frac{1}{2}}$

$: (4LM^2-LN^2)^{\frac{1}{2}} = (LQ^2-QS^2)^{\frac{1}{2}} : (LP^2-PR^2)^{\frac{1}{2}} = SL : LR$

$= AL : LR.$

11561. (Professor Rebuffel.)—Les arêtes AB, AD, AA' et la diagonale AC' d'un parallélipipède ABCD, A'B'C'D' sont rencontrées par une sphère qui passe par le sommet A aux points β, δ, α', γ'. Prouver que

(1) $AB \cdot A\beta + AD \cdot A\delta + AA' \cdot A\alpha' = AC' \cdot A\gamma'$,

et (2) que le lieu des centres des sphères qui passent par le sommet O d'un angle trièdre, et qui rencontrent les arêtes en des points A, B, C tels que $OA + OB + OC = l$, est un plan.

Solution by T. W. K. Clarke; Prof. Chakrivarti, M.A.; *and others.*

1. Let the sphere meet AB' in β'. Now $A\beta\beta'\alpha'$ are cyclic; therefore

$\angle \beta\alpha'\beta' = \angle \beta A\beta' = \angle AB'A'$,

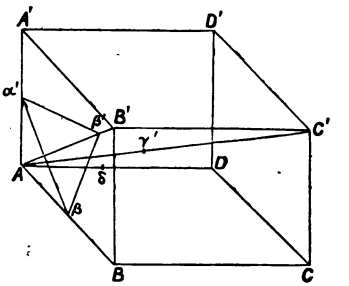
and $\angle \alpha'\beta\beta' = \angle \alpha' AB'$;

therefore triangles $\beta\alpha'\beta'$, AB'A' are similar; therefore

$$\frac{\beta\alpha'}{AB} = \frac{\alpha'\beta'}{B'A'} = \frac{'\ \beta}{A'A'};$$

therefore, from Ptolemy's theorem, we get

$AB' \cdot A\beta' = AB \cdot A\beta + AA' \cdot A\alpha'$.



Similarly, we have

$AC' \cdot A\gamma' = AB' \cdot A\beta' + AD \cdot A\delta = AB \cdot A\beta + AA' \cdot A\alpha' + AD \cdot A\delta$.

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2. If along OA, OB, OC, we take lengths, each $= r$, for the sides of a parallelepiped, we shall have, from (1), $r(OA+OB+OC) = OR \cdot OP$, where OR is the diagonal through O, and P the point where the sphere cuts it again.

Therefore, if $OA+OB+OC = \text{const.}$, OP is const., *i.e.*. P is fixed, and the locus of the centre of the sphere is the plane bisecting OP at right angles.

7674. (Professor HUDSON, M.A.)—From the top of a hill the depression of a point on the plain below is 30°, and from a spot three-quarters of the way down the depression of the same point is 15°; find within 1′ the inclination of the hill.

Solution by H. J. WOODALL; Professor BHATTACHARYA; *and others.*

Put $\tan CAE = X$, $AE = a$, $AP = x$;

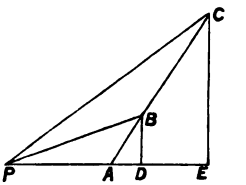
$\therefore\ aX/(a+x) = \tan 30°$,

$\frac{1}{4}aX/(\frac{1}{4}a+x) = \tan 15° = aX/(a+4x)$;

$\therefore\ \frac{1}{3}\sqrt{3} = aX/(a+x) = 2\tan 15°$

$= 2aX(a+4x)/\{(a+4x)^2 - a^2X^2\}$;

and $aX\sqrt{3} = a+x$; $\therefore\ x = a(X\sqrt{3}-1)$.



Substitute in second equation, and we get

$$23X^2 - 18\sqrt{3}\,.\,X + 9 = 0.$$

Hence we obtain $X = \{9\sqrt{3} \pm (243-207)^{\frac{1}{2}}\}/23$,

$X = (9\sqrt{3}\pm 6)/23 = 9{\cdot}58872/23$ or $21{\cdot}58872/23$

$= 0{\cdot}416904$ or $= 0{\cdot}93864$

$= \tan 22°\ 37'\ 52'' + \ldots$ or $= 43°\ 11'\ 14'' + \&c.$

[Prof. HUDSON solves the problem thus:—Let $PA = x$, $CAE = a$; then

$PC = 3x$, $CE = 3x/2$, $PE = 3\sqrt{3}x/2$, $AE = x(3\sqrt{3}-2)/2$,

$\tan a = 3(3\sqrt{3}-2) = (9\sqrt{3}+6)/23 = {\cdot}93864$;

therefore CAE is nearly $\frac{1}{4}\pi$. For a closer approximation, we have

$\tan(\frac{1}{4}\pi - a) = (1-\tan a)/(1+\tan a) = {\cdot}06136/1{\cdot}93864 = {\cdot}0316.$

Since $({\cdot}0316)^3/3$ radians are less than 1′, we may take the tangent for the circular measure; therefore

$45° - CAE = 180 \times 60 \times {\cdot}0316 \div \pi$ minutes

$= 108 \times 3{\cdot}16 \ldots \div 3{\cdot}14 \ldots = 109' = 1°49';$

$\therefore\ CAE = 45° - 1°49 = 43°11'.$]

5966. (ELIZABETH BLACKWOOD.) — Find the average area of the circle that passes through three points taken at random on the surface of a sphere.

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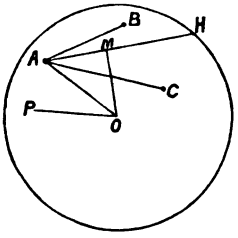
Solution by Professor G. B. M. ZERR.

Let AH be the diameter of the section of the sphere made by a plane through the three random points A, B, C; M its centre; O the centre of the sphere; OP a line such that AB is parallel to the plane MOP.



Let $AO = r$, $\angle AOM = \theta$, $\angle BAM = \phi$, $\angle CAM = \psi$, $\angle MOP = \lambda$, and the angle the plane POM makes with a fixed plane through $OP = \rho$. Then we have

$$AB = 2r \sin\theta \cos\phi,$$

$$AC = 2r \sin\theta \cos\psi.$$

Then Area circle $= \pi r^2 \sin^2\theta$.

An element of surface is, at A, $4\pi r^2 \sin\theta\, d\theta$; at B, $4r^2 \sin\theta \cos\phi\, d\phi\, d\lambda$; at C, $4r^2 \sin\theta \sin(\phi+\psi) \sin\lambda \cos\psi\, d\psi\, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π.

Since $16\pi^3 r^6$ is the whole number of ways the three points can be taken, we get, for the required average,

$$\Delta = \frac{1}{\pi^3 r^6} \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} \int_0^{\pi} \int_0^{2\pi} \pi r^2 \sin^2\theta \cdot 4\pi r^2 \sin\theta\, d\theta \cdot 4r^2 \sin\theta \cos\phi\, d\phi\, d\lambda \times 4r^2 \sin\theta \sin(\phi+\psi) \sin\lambda \cos\psi\, d\psi\, d\rho$$

$$= 2r^2 \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} \int_0^{\pi} \sin^5\theta \sin(\phi+\psi) \cos\phi \cos\psi \sin\lambda\, d\theta\, d\phi\, d\psi\, d\lambda$$

$$= 4r^2 \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\phi}^{\frac{1}{2}\pi} \sin^5\theta \sin(\phi+\psi) \cos\phi \cos\psi\, d\theta\, d\phi\, d\psi$$

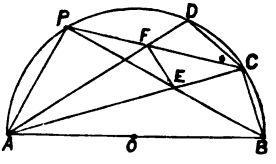
$$= r^2 \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^5\theta \left\{(\pi + 2\phi) \sin\phi \cos\phi + 2\cos^2\phi\right\} d\theta\, d\phi$$

$$= \frac{3\pi r^2}{2} \int_0^{\frac{1}{2}\pi} \sin^5\theta\, d\theta = \tfrac{4}{5}\pi r^2$$

11458. (EDITOR.)—If P be any point in the circumference, and BC, CD two equal arcs, of a semicircle APB; then, if CA, PB meet in E, and AD, PC in F, prove that AD is perpendicular to EF.

Solution by T. SAVAGE, M.A.; M. BRIERLEY; *and others.*

The angles BAC, CAD, EPF are equal, as standing on equal arcs BC, CD. Hence A, P, F, E are concyclic; and angle AFE = angle APE, which is a right angle.



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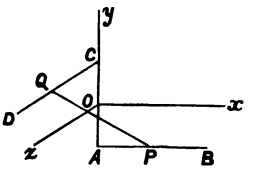
11465. (H. Brocard.)—On donne deux droites rectangulaires AB et CD, non situées dans un même plan; soit AC leur plus courte distance. Une droite s'appuie sur AB et CD en interceptant des segments égaux à partir de A et C. Déterminer la courbe d'intersection de la surface engendrée par cette droite avec la sphère décrite sur AC comme diamètre.

Solution by H. W. Curjel, B.A.; Professor Bhattacharya; *and others.*

Take the origin at O the middle point of AC and OC as axis of y, and axes of x and z parallel to AB and CD respectively.

Let $OC = b$. Equation to PQ cutting intercepts AP, CQ, each $= a$, from AB and CD is

$$\frac{x-\frac{1}{2}a}{-a} = \frac{y}{2b} = \frac{z-\frac{1}{2}a}{a};$$



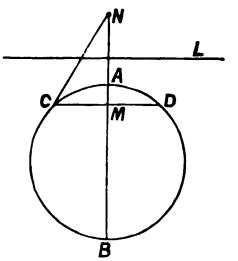
therefore surface generated by PQ as a varies is $(x+z)\,y/b = z-x$; therefore curve in which this surface cuts the sphere on AC as diameter is $x^2+y^2+z^2 = b^2$; and is therefore on the cone $(x^2+z^2)^2 = 2zx(x^2+2y^2+z^2)$.

11350. (Belle Easton, B.Sc.)—AB is the vertical diameter of a fine circular tube in which move three equal particles P, Q, Q′ (modulus of restitution = 1 for any pair); P starts from A, and Q, Q′ in opposite senses from B, with such velocities that at the first impact all three have equal velocities; prove that throughout the motion the line joining any pair is either horizontal or passes through one of two fixed points, and that the intervals of time between successive impacts are all equal.

Solution by H. W. Curjel, B.A.; Professor Cavallin; *and others.*

Since at the time of the first impact the velocities are equal and the particles in the same horizontal line, the particles must have been projected with velocities due to their depths below some fixed horizontal line, L say.

Let M and N be the limiting points of the coaxal system S of circles determined by the given circle ACBD and L (L being radical axis of system). P may be considered as having been projected with the same velocity as Q′ from B, half the time of a complete revolution before Q′.



Therefore, before the first impact, PQ′ always touches some circle of system S (Tait and Steele's *Dynamics*, Chap. vi., Ex. 9). But initially PQ′ passes through M and N; therefore this circle is one of the limiting points M, N; it is evidently M, for PQ′ cuts AB

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inside the circle. Hence also QP passes through N, for QQ′ is perpendicular to MN, and M, N are inverse points with respect to the circle.

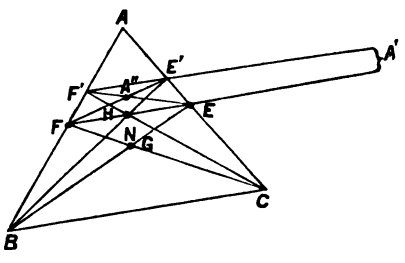
Hence the first impact will take place at C, one of the points of contact C, D of tangents from N, Q′ being at D; let this take place at time t from the beginning of the motion. Since the velocities and momenta of the impinging particles are equal and opposite, their velocities will be reversed by the impact. Hence, after another interval t, Q will be at B, and P, Q′ impinge at A, having their velocities reversed by the impact, and so on.

It is also evident that the relative positions of any two of the particles at any time is the same as or a reflexion with respect to AB of some position of the particles before the first impact; hence the line joining any two particles is either horizontal or passes through M or N.

11377. (Professor Neuberg.)—Soit A′B′C′ le triangle formé par les tangentes aux points A, B, C de l'hyperbole de Kiepert du triangle ABC. Démontrer (1) que A′, B′, C′ sont les intersections des droites joignant les milieux des côtés de ABC, avec les droites joignant les pieds des hauteurs de ABC; (2) que les droites AA′, BB, CC′ sont perpendiculaires à la droite joignant le centre de gravité de ABC à l'orthocentre; (3) que l'orthocentre du triangle A′B′C′ coincide avec le centre du cercle des neuf points de ABC.

Solution by Professors Ramaswami Aiyar, Chakrivarti, *and others.*

Let G be the centroid, H the orthocentre, and N the N.P. centre; D, E, F the middle-points of the sides; D′, E′, F′ the feet of the perpendiculars. Suppose that EF, E′F′ intersect at a point A′, and E′F, EF′ at A″, &c.



By Pascal's Theorem, applied to the hexagon BEF′CFE′, the points G, H, A″ are collinear. And the triangle AA′A″ is self-conjugate with respect to the N.P. circle; hence N is its orthocentre.

Therefore GH is perpendicular to AA′, and similarly to BB′, CC′ ..... (α)

The equations of EF, E′F′ are

$$-\alpha \sin A + \beta \sin B + \gamma \sin C = 0, \qquad -\alpha \cos A + \beta \cos B + \gamma \cos C = 0;$$

hence A′ is the point $\{-\sin(B-C),\ \sin(C-A),\ \sin(A-B)\}$.

By symmetry we know the points B′, C′; the line joining them is

$$\beta/\sin(C-A) + \gamma/\sin(A-B) = 0.$$

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Now this line is the tangent at A to the Kiepert's hyperbola

$$\sin(B-C)/\alpha+\sin(C-A)/\beta+\sin(A-B)/\gamma=0 \quad \ldots\ldots\ldots\ldots \quad (\beta).$$

It is also the harmonical conjugate of AA', whose equation is

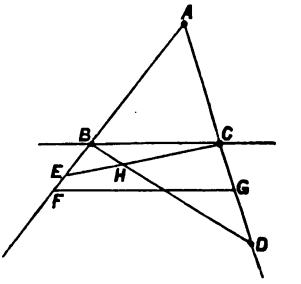
$$\beta/\sin(C-A)=\gamma/\sin(A-B);$$

hence AA'', B'C' is one line, and is the polar of A' with respect to the N.P. circle; thus we see that A'B'C' is a self-conjugate triangle; therefore N is the orthocentre of A'B'C' ..... (γ).

The parts (α), (β), and (γ) prove the theorems given.

11450. (Professor Bernès.)—Dans un triangle ABC, les perpendiculaires en B, C, à AB, AC, rencontrent AC, AB en D, E. Démontrer que, si une parallèle quelconque à BC rencontre AB, AC en F, G, l'axe radical des circonférences BGD, CFE passe par A et par le centre de la circonférence ABC.

Solution by R. F. Davis, M.A.; J. H. Grace; *and others.*



Let BD, CE intersect in H, so that AH is a diameter of the circumcircle of ABC. Then, since E, B, C, D are obviously concyclic, $BH.HD = CH.HE$

and $AB.AE = AC.AD.$

But $AB : AC = AF : AG;$

hence $AE.AF = AD.AG.$

Thus we have shown that A, H are both points on the radical axis of the circumcircles of BGD, CFE.

11549. (Professor W. H. S. Monck, M.A.)—Supposing the chance of a Knight occupying a particular square to be proportional to the number of squares from which that square might have been reached, find the mean range of the Knight.

Solution by H. W. Curjel, B.A.; Professor Aiyar; *and others.*

$$\text{Mean range} = \frac{\Sigma[(\text{range})^2 \text{ of a square}]}{\Sigma(\text{range of a square})};$$

the summation extending over the whole board

$$= \frac{4\times2^2+8\times3^2+16\times4^2+4\times4^2+16\times6^2+16\times8^2}{4\times2+8\times3+16\times4+4\times4+16\times6+16\times8} = \frac{251}{42}.$$

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The accompanying figure gives quarter of the board, with the ranges of the squares marked in the squares, the corner of board being at A,

A

2	3	4	4
3	4	6	6
4	6	8	8
4	6	8	8

10191. (Professor THARAUD.)—Étant données deux droites rectangulaires OX, OY, on prend sur OY les points A, B, et sur OX le point C, tels que $OA = mOB = mOC = a$, m étant un nombre positif quelconque et a une longueur donnée. (1) Trouver sur la droite indéfinie BC un point M tel que le rapport de ses distances au point A et à la droite OX ait une valeur donnée λ; (2) discuter en laissant m constant et faisant varier λ: étudier comment sont placés les points trouvés par rapport aux points B et C; (3) il y a en général deux points M et M′ satisfaisant au problème: déterminer λ de manière que ces points se confondent, et chercher entre les coordonnées du point Q correspondant à ce cas, une relation indépendante de m; (4) en partant de la relation précédente, construire le lieu décrit par le point Q lorsque la droite BC se déplace parallèlement à elle-même.

Solution by H. J. WOODALL.

(1) Take a point P in BC; draw its ordinate. Find

$$PD = PD' = \lambda . PR.$$

From A draw AM, AM′ parallel to DP, D′P respectively.

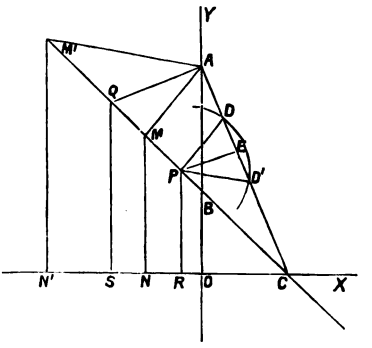
Draw the ordinates of M, M′.

Then $MN : PR = MC : PC = MA : PD$;

$\therefore MA : MN = PD : PR = \lambda : 1$.

Similarly,

$M'A : M'N' = PD' : PR = \lambda : 1$.



(2) Then, as λ varies, the points D, D′ will be wider apart or nearer as λ is larger or smaller.

(3) The above construction gives the positions of M and M′ satisfying the problem. When M coincides with M′ at Q, then AQ : QS = least possible value of λ. Returning to construction, at point P we

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find, in this case, PE perpendicular to AC; therefore QA is perpendicular to AC. Then, in (2), we find angles M'AQ = QAM.

By analytical conics, it can be found easily that the locus of Q (as m varies) is $x(x+y) = a(a-y)$.

To find asymptotes (geometrically). When BC is $x+y = -a$, AC is perpendicular to BC; therefore QA is parallel to BC; therefore asymptote in *direction* $x+y=0$. Again, when BC is $x+y = +\infty$, AC is $y = a$. Therefore QA meets BC at infinity; therefore asymptote in *direction* $x=0$.

11424. (G. E. Crawford, M.A.)—If ABC be a triangle, and AA', BB', CC' be three diameters of the circumscribing circle, prove that the sum of the projections of BC, CA, AB on AA', BB', CC', respectively, vanishes, if the proper convention of signs be adopted.

Solution by W. J. Greenstreet, M.A.; Professor Aiyar; *and others.*

The projection of BC on AA' is

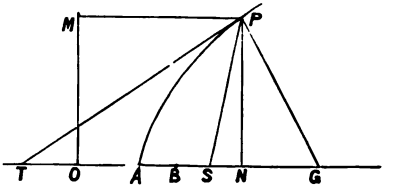
$$R(\cos\tfrac{1}{2}B - \cos\tfrac{1}{2}C), \quad \text{and} \quad \Sigma R(\cos\tfrac{1}{2}B - \cos\tfrac{1}{2}C) = 0.$$

[The trigonometrical equivalent is

$$a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0.]$$

7306. (By Professor Hudson, M.A.)—From a point P on a parabola, focus S, PM, PN are drawn perpendicular to the directrix and axis; PT, PG are the tangent and normal limited by the axis. What line represents the resultant of forces represented by PM, PT, PS, PN, PG?

Solution by H. J. Woodall.



Resultant of PM and PN is PO.

Resultant of PT and PG is 2PS.

Therefore 5 forces reduce to PO + 3PS = 4PB.

when OB = 3BS.

i.e., AB = BS.

11570. (E. M. Langley, M.A.)—AD, CF are the external common tangents to two circles ABC, DEF; BE one of the internal. The diameters through B and E cut AC and DF respectively in G and H. Show

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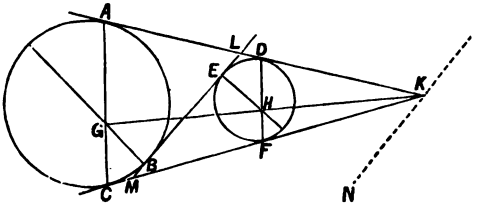
(1) that GH bisects BE; and find (2) whether the above theorem is also true if "external" and "internal" be interchanged.

Solution by T. W. K. Clarke; I. Arncld; *and others.*

1. Through K, the ex-centre of similitude, draw KN parallel to BE. Now the polar of H with respect to circle DEF must pass through K



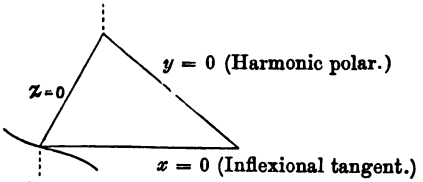
and must be perpendicular to EH, *i.e.*, KN is the polar of H with respect to the circle DEF. So also KN is the polar of G with respect to the circle ABC; therefore GH passes through K, and the pencil KD, KH, KF, KN is harmonic; therefore KHG bisects LM, which is parallel to KN, *i.e.*, GH bisects BE (since LE = BM).

2. An exactly similar proof holds if we exchange the words "internal" and "external."

5485. (R. A. Roberts, M.A.)—If tangents are drawn to a cubic from any point of a harmonic polar, prove that they form a pencil in involution.

Solution by Professor Schoute.

(Arbitrary point of harmonic polar.)



(Point of inflexion.)

In the coordinates indicated in the diagram, the equation of the cubic is

$$x(Ax^2 + By^2 + Cz^2 + 2Dxz) + z^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

By elimination of x between (1) and the first polar curve

$$3Ax^2 + By^2 + Cz^2 + 4Dxz = 0$$

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of the point $y = 0$, $z = 0$, we find

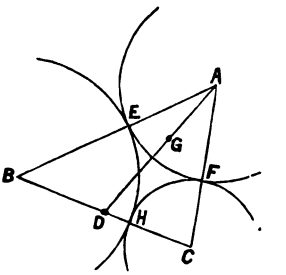
$$\begin{vmatrix} -2A, & -2Dz, & 0, & z^3, & 0 \\ 0, & -2A, & -2Dz, & 0, & z^3 \\ 3A, & 4Dz, & By^2+Cz^2, & 0, & 0 \\ 0, & 3A, & 4Dz, & By^2+Cz^2, & 0 \\ 0, & 0, & 3A, & 4Dz, & By^2+Cz^2 \end{vmatrix} = 0.$$

This equation of the six tangents is of the form $f(y^2, z^2) = 0$. This proves that these tangents are in involution; $y = 0$ and $z = 0$ are the double elements of this involution.

11278. (D. Biddle.)—The bases of three right cones, resting on a horizontal plane, touch each other, and are 300, 200, 100 square feet in area, whilst the respective heights are 100, 400, 300 feet (linear). A marble falls from a point above the centre of gravity of the three cones and one-third higher: find where it strikes the plane.

Solution by Professor Zerr, M.A.; *the* Proposer; *and others.*

Let B, A, C be the centres of the bases of 1st, 2nd, 3rd cones; then



$$BE = BH = 10\left(\frac{3}{\pi}\right)^{\frac{1}{2}};$$

$$AE = AF = 10\left(\frac{2}{\pi}\right)^{\frac{1}{2}};$$

$$CF = CH = 10\left(\frac{1}{\pi}\right)^{\frac{1}{2}};$$

$$AB = c = AE + BE = \frac{10}{\sqrt{\pi}}(\sqrt{3}+\sqrt{2});$$

$$AC = b = AF + CF = \frac{10}{\sqrt{\pi}}(\sqrt{2}+1);$$

$$BC = a = BH + CH = \frac{10}{\sqrt{\pi}}(\sqrt{3}+1).$$

Since D is the mid-point of BC,

$$AD = e = \tfrac{1}{2}\{2b^2+2c^2-a^2\}^{\frac{1}{2}} = \frac{5\sqrt{2}}{\sqrt{\pi}}\{2\sqrt{2}(1+\sqrt{3})+\sqrt{3}(2\sqrt{3}-1)\}^{\frac{1}{2}}.$$

Project the centre of gravity of each cone on the centre of its base. Then the centre of gravity of B and C is at D, the mid-point of CB, since the volume of each is 10000; and $46666\frac{2}{3} : 20000 = e : x$; therefore $x = \frac{3}{7}e$ = distance from A of projection of centre of gravity of the three cones on the plane. $\frac{3}{7}e = AG = \frac{15\sqrt{2}}{7\sqrt{\pi}}\{2\sqrt{2}(1+\sqrt{3})+\sqrt{3}(2\sqrt{3}-1)\}^{\frac{1}{2}}$.

The centres of gravity of (B) and (C) are respectively 25 and 75 feet above the plane; therefore their joint centre of gravity is 50 feet above D. And, as the centre of gravity of (A) is 100 feet above the plane, the

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joint centre of gravity of the three cones is $78\frac{4}{7}$ feet above G, viz., $100-\frac{3}{7}(100-50)$. Consequently, the point of rest of the marble is $\frac{4}{3}(78\frac{4}{7})$, or $104\frac{16}{21}$ feet above G, which $=\frac{11}{12}$ of the height of the cone (A). The radius of (A) at the base is 7·991369 feet, and $\frac{3}{4}\frac{1}{3}$ of this is 5·898391 = the radius of the cone (A) at the level of the aforesaid point of rest. The value of AG given above is ·742164 of the radius of (A) at the base, = 5·9309064 feet. Consequently, the marble, which we must regard as a material point or particle, is horizontally distant from the cone only ·032515 of a foot, or ·390180 of an inch. This, however, gives it a fall of 19·53 inches before it impinges on the cone for the first time. Consequently, the point of first impact is 103·134405 feet from the plane, or, measured on the surface of the cone, 103·154983 feet, the angle between the axis and generating line of the cone being $1°\,8'\,39·394''$ ($\equiv\alpha$). Now, it can easily be proved that the succeeding impacts occur at points which are the projections on the surface of the cone of the several positions which would be reached at intervals of $2t$ (where t = the time occupied, before the first impact, in falling from rest), if the particle were not intercepted by the cone. Thus, let $p = 19·53$ inches = 1·6275 feet. Then, the nth impact occurs at $8p\,.\cos\alpha\{1+2+\ldots+(n-1)\}$ from the point of first impact, measured on the surface of the cone. And as $8p\,.\cos\alpha = 13·017406$ feet, the successive impacts occur at 13·017406, 39·052218, and 78·104436 feet from the first. Thus the last impact on the cone occurs at a point 25·050547 feet from the foot of the cone. The length of the next interval, if the cone extended further down, would be 52·069624. Let $2tn$ = the time occupied between the last impact on the cone and the striking of the plane. Then

$$4p(m-m^2) = (52·069624m-25·050547)\cos\alpha,$$

hence $$m = 0·512443.$$

Moreover, let x = the distance (on AD) beyond the cone at which the particle strikes the plane. Then $x = 4p(m-m^2)\tan\alpha$,

or $$x = (52·069624m-25·050547)\sin\alpha$$
$$= 1·621527\sin\alpha = 0·032381 \text{ of a foot, or } ·388572 \text{ of an inch.}$$

rather less than its horizontal distance from the cone at starting.

11357. (G. E. Crawford, B.A.)—If $\dfrac{az+b}{cz+d} = u\equiv\xi+i\eta$ and $z\equiv x+iy$, prove that, if (x, y) describes a circle, so does (ξ, η), the four constants being all real.

Solution by H. W. Curjel, B.A.

Let (x, y) describe the circle, $x^2+y^2 = L$, a linear function of x and y, and let $c^2(x^2+y^2)+2cdx+d^2 = v$; then

$$\xi+i\eta = \frac{ax+b+iay}{cx+d+icy} = \frac{ac(x^2+y^2)+cbx+adx+bd+i(ad-cb)y}{v};$$

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therefore $\xi = \dfrac{ac\mathrm{L} + (cb+ad)\,x + bd}{v}$, $\eta = \dfrac{(ad-cb)\,y}{v}$;

hence x/b and y/b can be expressed as linear functions of ξ and η; thus

$$\xi^2+\eta^2 = \frac{[a^2(x^2+y^2)+2abx+b^2]\,v}{v^2} = \frac{a^2\mathrm{L}+2abx+b^2}{v}$$

$$= \frac{\text{a linear function of } x \text{ and } y}{v} = \text{a linear function of } \xi \text{ and } \eta;$$

therefore (ξ, η) describes a circle.

11109. (A. Gordon.)—If

$$\mathrm{Q}_{r+2} - (a+b)\,\mathrm{Q}_{r+1} + ab\mathrm{Q}_r = 0, \quad ab\,.\,\mathrm{P}_{r+2} - (a+b)\,\mathrm{P}_{r+1} + \mathrm{P}_r = 0,$$

and $$\mathrm{P}_0 = \mathrm{Q}_0 = 0, \quad \mathrm{P}_1 = \mathrm{Q}_1 = 1;$$

prove that $\mathrm{P}_1\mathrm{Q}_1 - 3\mathrm{P}_3\mathrm{Q}_3 + 5\mathrm{P}_5\mathrm{Q}_5 \ldots (-1)^m (2m-1)\,\mathrm{P}_{2m-1}\mathrm{Q}_{2m-1}$ is

$$\frac{ab\,(-1)^m}{(a-b)^2}\left\{2m - 2m\,\frac{a^{4m}+b^{4m}}{(a^2+b^2)(ab)^{2m-1}} + \frac{(a^2-b^2)(a^{4m}-b^{4m})}{(a^2+b^2)^2\,(ab)^{2m-1}}\right\}.$$

Solution by H. W. Curjel, B.A.

$$\mathrm{P}_r = \frac{ab}{a-b}\left\{\left(\frac{1}{b}\right)^r - \left(\frac{1}{a}\right)^r\right\}, \quad \mathrm{Q}_r = \frac{a^r-b^r}{(a-b)};$$

$$\mathrm{P}_r\mathrm{Q}_r = \frac{ab}{(a-b)^2}\left\{\frac{a^r}{b^r} + \frac{b^r}{a^r} - 2\right\};$$

therefore $\mathrm{P}_1\mathrm{Q}_3 - 3\mathrm{P}_3\mathrm{Q}_3 + 5\mathrm{P}_5\mathrm{Q}_5 - \ldots (-1)^m (2m-1)\,\mathrm{P}_{2m-1}\mathrm{Q}_{2m-1}$

$$= \frac{ab\,(-1)^m}{(a-b)^2}\left\{2m - 2m\,\frac{a^{4m}+b^{4m}}{(a^2+b^2)\,(ab)^{2m-1}} + \frac{(a^2-b^2)(a^{4m}-b^{4m})}{(a^2+b^2)^2\,(ab)^{2m-1}}\right\},$$

for $1 + 3x + 5x^2 + \ldots + (2n-1)\,x^{m-1} = \dfrac{1}{1-x} + 2x\,\dfrac{1-x^{m-1}}{(1-x)^2} - (2m-1)\,\dfrac{x^n}{1-x}$.

11538. (R. Tucker, M.A.)—The image of a parabola with respect to the tangent at P $(am^{-2}, 2am^{-1})$ is the parabola

$$\left[m\,(m^2-1)\,\mathrm{Y} + 2m^2\mathrm{X} + 2a\right]^2 + 4am^2\,(m^2+1)\left[\mathrm{X}\,(m^2-1) - 2m\mathrm{Y} + 2a\right] = 0.$$

Find the locus of the vertices and the envelope of the tangents at them for different positions of P.

Solution by Rev. T. Galliers, M.A.; R. Knowles, B.A.; *and others.*

Let $p\,(a\mu^{-2}, 2a\mu^{-1})$ be any point on the parabola; then, if (x, y) be the image of p, we have

$$(\mu y - 2a)/\mu = (a-\mu^2 x)/(\mu^2 m) = 2\,(m-\mu)^2\,a/\left\{\mu^2 m\,(1+m^2)\right\},$$

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whence $(my+x)\,\mu^2-2am\mu = a$ .....(1),

$\{(1+m^2)\,x+2a\}\,\mu^2-4am\mu+a\,(m^2-1) = 0$ .....(2).

Eliminating μ between (1), (2), we get, for the equation of the image of the parabola,

$$\left[m\,(m^2-1)\,y+2m^2x+2a\right]^2+4am^2\,(m^2+1)\left[(m^2-1)\,x-2my+2a\right] = 0.$$

This represents a parabola, of which the axis and the tangent at the vertex have for their equations, respectively,

$$2m^2.\,x+m\,(m^2-1)\,y+2a = 0, \quad (m^2-1)\,x-2my+2a = 0.$$

Hence the locus of the vertex for the different positions of P, and the envelope of the tangent at the vertex, are

$x^3+y^2(x+2a) = 0$, a *cissoid*, and $x^2+y^2-2ax = 0$, a *circle*.

11486. (The late Professor WOLSTENHOLME, Sc.D.)—A rod AB is marked at random in P, and points Q, R are then taken at random in AP, PB, respectively; prove that (1) the chance that the sum of the squares on AQ, RB will exceed the sum of those on QP, PR is $\frac{1}{2}$; but (2), when Q. R are first taken at random in AB, and P then taken at random in QR, the chance of the same event is $\frac{3}{8}\,(3-2\log 2)$.

Solution by Professor ZERR.

1. Let AB = unity, AP = x,
AQ = y, BR = z;

then, from $y^2+z^2 = (x-y)^2+(1-x-z)^2$,

$$z = (1+2x^2-2xy-2x)/2\,(1-x);$$

$$\therefore\quad p_1 = \int_0^1\int_0^x \left\{(1+2x^2-2xy-2x)/2\,(1-x)\right\}\,dx\,dy + \int_0^1\int_0^x (1-x)\,dx\,dy$$

$$= 3\int_0^1 (x-x^2)\,dx = \tfrac{1}{2}.$$

2. Let AR = y, AQ = x, PR = z; then from

$$x^2+(1-y)^2 = z^2+(y-x-z)^2, \quad z = \tfrac{1}{2}\left[x-y-\left\{2-4y+(x+y)^2\right\}^{\frac{1}{2}}\right];$$

$$\therefore\quad p_2 = \tfrac{1}{2}\int_0^1\int_0^y \left[x-y-\left\{2-4y+(x+y)^2\right\}^{\frac{1}{2}}\right]dx\,dy \div \int_0^1\int_0^y (x-y)\,dy\,dx$$

$$= \tfrac{3}{2}\int_0^1 \left\{ y^2+y\,(2-4y+y^2)^{\frac{1}{2}}-2y\,(2-4y+4y^2)^{\frac{1}{2}} \right.$$

$$\left. +(2-4y)\log\left[\frac{y+(2-4y+y^2)^{\frac{1}{2}}}{2y+(2-4y+4y^2)^{\frac{1}{2}}}\right]\right\}dy = \tfrac{3}{8}\,(3-2\log 2).$$

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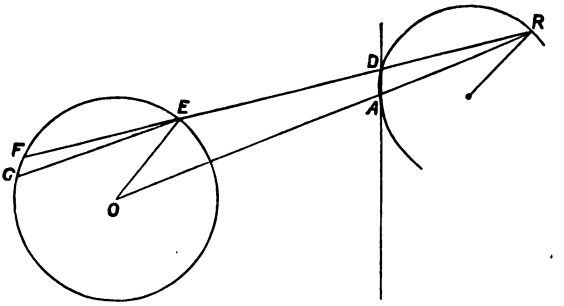
7520. (The Editor.)—Through a given point R, draw, geometrically, a straight line RDEF, cutting a straight line in D, and a given circle in E, F, so that RD may be equal to EF.

Solution by Morgan Brierley.

Let R be the given point, DA the given straight line, and O the centre of the given circle. Join R, O, cutting DA in A; parallel to RO



draw the chord CE = AR; with radius = OE. draw a circle through R, A, cutting the line DA in D; and through R, D, E, draw RDEF; then this will be the line required.

Manifestly, FE = RD, both being chords equi-distant from the centres of equal circles.

[This problem arose out of the Editor's Question 7462. At the end of the solution of this question which is given on pp. 99, 100 of Vol. 40, an *algebraic* solution of Quest. 7520 is given. See also Vol. 43, page 120, where a simpler solution of Quest. 7462 is given.]

11505. (D. Biddle.)—A marble falls from a given height h above a horizontal plane, but is intercepted at $\frac{1}{2}h$ by a small inclined plane, from which it rebounds. The angle of this small plane is constantly varying between 0° and 90°. Find the mean distance from the vertical at which the marble strikes the horizontal plane.

Solution by Professor G. B. M. Zerr.

Let e be the coefficient of restitution, β the inclination of the small plane; then x = horizontal range

$$= e^2h \sin 2\beta \cos 2\beta \pm eh \sin 2\beta \,(1 + e^2 \cos^2 2\beta)^{\frac{1}{2}};$$

therefore $\Delta = eh \int_0^{\frac{1}{4}\pi} \{e \sin 2\beta \cos 2\beta + \sin 2\beta \,(1 + e^2 \cos^2 2\beta)^{\frac{1}{2}}\} d\beta \Big/ \int_0^{\frac{1}{4}\pi} d\beta$

$$= h/\pi \log \{(1 + e^2)^{\frac{1}{2}} - e\}.$$

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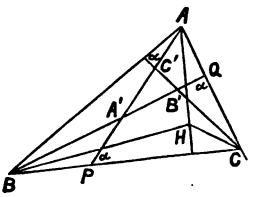
10958. (Professor DE LONGCHAMPS.)—Soient AB, BC, CA les directions positives des côtés d'un triangle ABC. Par les sommets, on mène des droites AP, BQ, CR, faisant, avec ces directions positives, un même angle α. Les droites considérées forment un triangle A'B'C'; démontrer que (1) le centre du cercle circonscrit à A'B'C' coincide avec l'orthocentre de ABC; (2) tout point remarquable de A'B'C' décrit une circonférence. Toutes ces circonférences passent par un même point, l'orthocentre de ABC. Elles constituent un *réseau* de circonférences remarquables relativement au triangle ABC.

Solution by W. J. GREENSTREET, M.A.



If the perpendiculars AH, BH, CH be turned through an angle $\frac{1}{2}\pi - \alpha$, they coincide with AP, BQ, CR, respectively.

Therefore the point A' lies upon the circumcircle of the triangle BHC, and similarly the vertices B' and C' are on the same circle.

Therefore the triangles ABC and A'B'C' are directly similar, and H is the centre of similitude of all triangles such as A'B'C'.

When $\alpha = 0$, H is the circumcentre of the triangle A'B'C'.

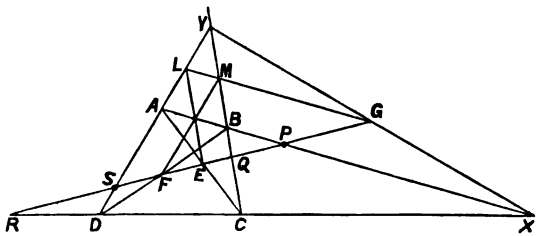
If P be the in-centre (say) of the triangle A'B'C', the triangle PA'H remains similar to itself; and the points A', P describe two similar figures having their double point at H.

Therefore the locus of P is a circle through H, and similarly for any other remarkable point.

11540. (J. H. GRACE.)—E and F are the mid-points of the diagonals (internal) of a quadrilateral. The line EF meets the third diagonal in G, and the sides of the quadrilateral in P, Q, R, and S, respectively (P and R being on opposite sides). Prove that

$$GE \cdot GF = GP \cdot GR = GQ \cdot GS.$$

Solution by J. RICE; R. KNOWLES, B.A.; *and others.*



Let X, Y be extremities of third diagonal.

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Let L be mid-point of YA, M of YB.

Then L, M, G are collinear, and LE is parallel to YC, and MF is parallel to YD. Hence we have $GQ : GE = GM : GL = GF : GS$;

$$GQ \cdot GS = GE \cdot GF = GP \cdot GR.$$

11562. (Professor MALET, F.R.S.)—If a right line be drawn cutting a conic U in A and B, and a conic V in C and D, prove that the tangents to U at A and B intersect the tangents to V at C and D in four points which lie on a conic passing through the four points of intersection of U and V.

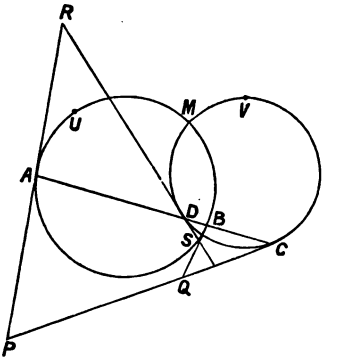
Solution by T. W. K. CLARKE; J. C. ST. CLAIR; *and others.*

Let U, V cut in K, L, M, N. Let tangent at C cut tangents at A and B in P and Q, and tangent at D cut them at R and S. Project the figure so that K, L become the circules; then the conics become circles cutting in M and N.

$$\angle RSQ = SDB + SBD$$
$$= PCA + PAC;$$
$$\therefore \angle RSQ + RPQ$$
$$= PCA + PAC + APC$$
$$= 2 \text{ right angles};$$

$\therefore$ PQ, RS are concyclic.

That is, in the original figure, PQRSKL lie on a conic.



Similarly, by projecting K, M into the circules, PQRSKM are on a conic, and also by projecting K, N into the circules, PQRSKN are on a conic; but these three conics are the same, since they have five points PQRSK common; therefore PQRSKLMN lie on a conic.

[Prof. SCHOUTE gives the following solution of the problem:—

Let $\epsilon = 0$ be the equation of the right line; and $\alpha = 0$, $\beta = 0$ those of the tangents in A and B to U; $\gamma = 0$, $\delta = 0$ those of the tangents in C and D to V. Then the equations of U and V are $\alpha\beta = \kappa_1\epsilon^2$, and $\gamma\delta = \kappa_2\epsilon^2$. The elimination of ϵ gives the equation $\kappa_2\alpha\beta = \kappa_1\gamma\delta$ of a conic passing through the four points common to U and V and the four points common to the couples α, β and γ, δ of right lines.]

11376. (Professor CURTIS, M.A.)—Any two chords of an equiangular hyperbola cutting each other perpendicularly have the rectangles under

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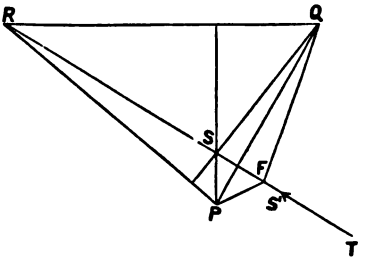
their sections equal and opposite in sign. Prove this, and derive the following:—An equiangular hyperbola and a circle, osculating, cut off equal and opposite portions from their common normal. [This latter proposition, we believe, is to be found in the German translation of SALMON's *Conics*. The above is a geometrical proof of it.]

Solution by H. W. CURJEL, B.A.



Let PQ, RS be the two chords intersecting in F; then, since two conics, the rect. hyp. and the line pair RS, PQ through RSPQ, are rect. hyperbolæ, all conics through PQRS are rect. hyperbolæ; hence PS is perpendicular to RQ, and QS to PR; thus the angles QRP, QSP = 2 right angles.

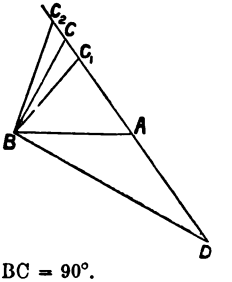
But if we produce SF to S', making S'F = FS, triangles PSQ, PS'Q will be equal in all respects, and therefore RPS'Q are concyclic.

Therefore $FP \cdot FQ = FS' \cdot FR = -FS \cdot FR$.

Again, if we take S as the end of the diameter conjugate to PQ, and move PQ parallel to itself till it become a tangent, the circle PSQ becomes the osculating circle at S. And if RS cuts the osculating circle again in T, $FS \cdot FT = FP \cdot FQ = -FS \cdot FR$; therefore $FT = -FR$. But in the limit F and S coincide; therefore the hyperbola and circle cut off equal and opposite portions of the common normal.

1196. (The EDITOR.)—Given the vertical angle and *one* of the containing sides, construct the triangle, when the ratio of the base to the sum of the other side and a given line is given, or a minimum.

Solution by H. J. WOODALL, A.R.C.S.



Here we have side AB and angle BAC given. Produce CA backwards, and make AD equal the given line. Join BD. Then in the triangle BDC we have

BC : DC = a given ratio = $1 : \lambda$

= sin BDC : sin DBC.

The triangle is solved when the angle DBC is obtained from this equation, viz., from

$$\sin DBC = \lambda \cdot \sin BDC.$$

The sum of the two values (DBC_1, DBC_2) of DBC is 180°, as is known. The minimum value of the ratio is when λ is greatest, that is, when DBC = 90°.

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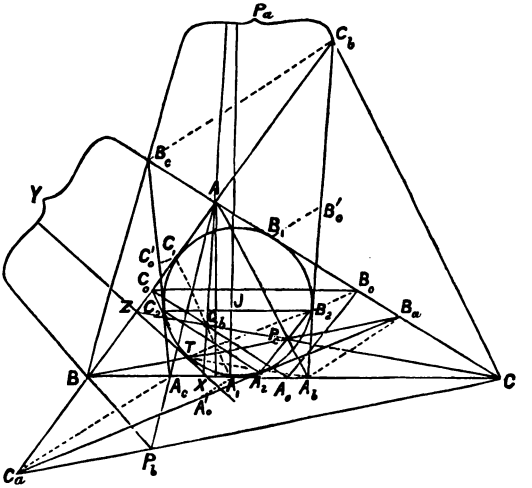
11563. (Professor LANGE, Ph.D.)—(1) Let A_0, B_0, C_0 be the mid-points of the sides BC, CA, AB of the triangle ABC, J the centre, and A_1, B_1, C_1 the points of contact of the inscribed circle with the sides. Turn the triangle round each of the lines AJ, BJ, CJ so as to have it successively in the positions AB_aC_a, BA_bC_b, CA_cB_c, while A_1, B_1, C_1 coincide with the points A_2, B_2, C_2 of the inscribed circle. Then A_0A_2, B_0B_2, C_0C_2 will meet in one point T, and this point proves to be the exterior centre of similitude of the inscribed circle J, and Feuerbach's circle F, and to be a common point of them. Hence J and F touch in T.

(2) Let BB_c, CC_b meet in P_a; AA_c, CC_a in P_b; AA_b, BB_a in P_c; AP_a, BC in X; BP_b, AC in Y; CP_c, AB in Z; then X, Y, Z lie in a straight line, viz., the common tangent of F and J in T.

Solution by Professor LAMPE, Ph.D.

(1) $A_1A_b = A_1A_c = A_cC_2 = A_bB_2 = AB_1$, and angle $A_b = A_c = A$; therefore B_2C_2 is parallel to A_bA_c parallel to B_0C_0; in the same way



A_2B_2 is parallel to A_0B_0, A_2C_2 parallel to A_0C_0. Hence $A_0B_0C_0$ and $A_2B_2C_2$ similar and similarly placed with respect to the exterior centre of similitude T. Moreover $AA_c : BB_c = AC : B_cC = AC_a : BC_b = AC : BC$, and angle $A_cAC_a = B_cBC_b$, whence $\triangle AA_cC_a \sim BB_cC_b$ and A_cC_a is parallel to B_cC_b: by the same reasoning A_cC_a is parallel to A_bB_a. Let A'_0, B'_0, C'_0 be the mid-points of B_aC_a, A_bC_b, A_cB_c; then $A_1A'_0$ is parallel to A_bB_a parallel to $B_1B'_0$ parallel to $C_1C'_0$, on account of $A_1A_b : A_1A_c = A'_0B_a : A'_0C_a = 1 : 1$. Since A_1 and A_2, A_0 and A'_0 are symmetrical points for AJ, we conclude angle $A_1A_0A_2 = A_1A'_0A_2 = A_cC_aB_a$; moreover, $BA_0C_0 = ACB = AC_aB_a$; therefore $TA_0C_0 = AC_aA_c$. Similarly, $TB_0C_0 = BC_bB_c = AC_aA_c = TA_0C_0$; therefore T is a point of the circle through $A_0B_0C_0$,

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and on account of $TA_2C_2 = TB_2C_2$ of the circle through $A_2B_2C_2$. $A_0B_0C_0$ being similar to $A_2B_2C_2$, T is the centre of similitude of J and F.

(2) BJ is perpendicular to CC_b, CI is perpendicular to BB_c; therefore P_aJ is perpendicular to BC, or P_aJA_1 a straight line. Let AA_3 be a perpendicular in ABC, then BCP_a and A_cA_bA being similar, with X as centre of similitude, A_0 and A_1 are corresponding points as mid-points of sides, A_3 and A_1 as foot-points of the perpendiculars. Therefore $XA_0 : XA_1 = XA_1 : XA_3$; therefore $(XA_1)^2 = XA_0 . XA_3$, or X is a point of the radical axis of F and J; the same conclusion holds for Y and Z; therefore XYZ is the radical axis, or here common tangent of F and J.

[This Question is connected with FEUERBACH's theorem (Quest. 11021) recently treated by Mr. R. F. DAVIS, on p. 51 of Vol. LVI. See H. SCHOETER, *Mathematische Annalen*, Vol. VII., pp. 517–531; and Prof. LANGE, *Archiv der Math. und Phys.*, Vol. III. (2nd Series), pp. 329–330.]

11591. (Professor LUCAS.)—De combien de manières un pion du jeu de dame, placé en un coin du damier, peut-il se rendre sur le bord opposé, en progressant par cases consécutives parallèlement à l'une ou l'autre diagonale? Le damier a $2n$ ou $2n+1$ cases par côté.

Solution by H. W. CURJEL, B.A.; J. C. ST. CLAIR; *and others.*

The piece starts from the corner A, and the numbers in the squares denote the number of ways they may be reached. Let u_m = the number

C D

1		12		65		208		429		572		429	
	1		11		54		154		275		297		132
		1		10		44		110		165		132	
			1		9		35		75		90		42
				1		8		27		48		42	
					1		7		20		28		14
						1		6		14		14	
							1		5		9		5
								1		4		5	
									1		3		2
										1		2	
											1		1
												1	
													0

B A

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of ways of crossing a board of m^2 squares. Then u_m = the sum of the numbers in the last row (CD).

The number in each square evidently = the sum of the numbers in the squares or square from which it may immediately be reached. Considering the rows parallel to the diagonal AC, we see that

$$u_m = 1 + (m-2) + \left\{\frac{(m-2)\,(m-3)}{2\,!} - 1\right\} + \left\{\frac{(m-2)\,(m-3)\,(m-4)}{3\,!} - (m-2)\right\}$$

$$+ \left\{\frac{(m-2)\,(m-3)\,(m-4)\,(m-5)}{4\,!} - \frac{(m-2)\,(m-3)}{2\,!}\right\} + \text{\&c. to } \frac{m}{2} \text{ or } \frac{m+1}{2} \text{ terms}$$

as m is even or odd ;

$$\therefore \quad u_{2n} = \frac{(2n-2)\,!}{(n-2)\,!\,n\,!} + \frac{(2n-2)\,!}{(n-1)\,!\,(n-1)\,!} = \frac{(2n-1)\,!}{n\,!\,(n-1)\,!},$$

$$u_{2n+1} = \frac{(2n-1)\,!}{n\,!\,(n-1)\,!} + \frac{(2n-1)\,!}{n\,!\,(n-1)\,!} = 2\,\frac{(2n-1)\,!}{n\,!\,(n-1)\,!} = 2u_{2n},$$

as we may also see, since we may advance from each square in the $2n^{\text{th}}$ row in two ways (either to the right or to the left).

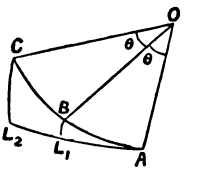
[Mr. St. Clair gave the same formula for u_m when m is even ; but his formula $u_m = C_{n-1}^{m-1}$ is true both for $m = 2n$ and $m = 2n-1$.

Example : $u_{11} = C_5^{10} = 252 = 2C_5^9 = 2u_{10}$.]

11516. (Professor Orchard, M.A., B.Sc.)—Show how the earth's radius may be found by sailing along a great circle, and taking latitudes.

Solution by H. J. Woodall, A.R.C.S ; Professor Ignacio Beyens ; *and others.*

Start from a point A on the equator, and determine the positions B and C, where AB = BC.



Then lat. B = BL_1 and lat. C = CL_2.

$$\text{Sin } CL_2 = \text{sin CA . sin CAL}_2,$$

$$\text{sin } BL_1 = \text{sin BA . sin BAL}_1.$$

Dividing, we get $\text{sin } CL_2 = 2 \cos BA \text{ . sin } BL_1$.

Now we have BL_1 and CL_2, whence we get arc AB = θ. But the length of AB is known = $a = r\theta$, whence r can be found.

8488. (Professor Orchard, B.Sc., M.A.)—Find the current given by a battery of 15 equal and similar cells in 5 rows of 5, 4, 3, 2, 1 cells respectively ; the electro-motive force of each cell being 1·6416, and the resistance of the wire joining the poles being the same as that of one cell.

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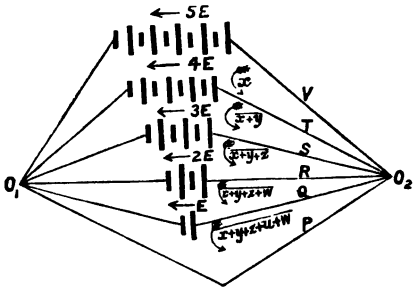
Solution by H. J. WOODALL, A.R.C.S.

This is solved by the application of MAXWELL'S Cyclic Equations (see Dr. FLEMING on Networks, *Phil. Mag.*, Sept., 1885).



The cyclic equations are:—

$$\left.\begin{aligned}
&(V+T)\,x - T\,(x+y) = E_5 - E_4 = E,\\
&(T+S)\,(x+y) - Tx - S\,(x+y+z) = E_4 - E_3 = E,\\
&(S+R)\,(x+y+z) - S\,(x+y) - R\,(x+y+z+w) = E_3 - E_2 = E,\\
&(R+Q)\,(x+y+z+w) - R\,(x+y+z) - Q\,(x+y+z+w+u) = E_2 - E_1 = E,\\
&(Q+P)\,(x+y+z+w+u) - Q\,(x+y+z+w) = E.
\end{aligned}\right\}$$

These simplify; also we have

$$V = 6r,\quad T = 5r,\quad S = 4r,\quad R = 3r,\quad Q = 2r,\quad P = r,$$

where r = resistance of one cell, and putting $E/r = K$;

$$\begin{aligned}
&\therefore\ E = Vx - yT; && \therefore\ K = 6x - 5y.\\
&\quad\ E = Ty - Sz; && \therefore\ K = 5y - 4z.\\
&\quad\ E = Sz - Rw; && \therefore\ K = 4z - 3w.\\
&\quad\ E = Rw - Qu; && \therefore\ K = 3w - 2u.\\
&E = P\,(x+y+z+w+u) + Qu; && \therefore\ K = x+y+z+w+3u.
\end{aligned}$$

Solving which, we get

$$x = \tfrac{29}{49}K,\quad y = \tfrac{25}{49}K,\quad z = \tfrac{19}{49}K,\quad w = \tfrac{9}{49}K,\quad u = -\tfrac{11}{49}K,$$

and

$$x + y + z + w + u = \tfrac{71}{49}K.$$

11382. (Professor TISSOT.)—Deux circonférences étant donnés ainsi qu'une droite qui ne les rencontre pas, on peut mener, d'un point quelconque de la droite, une tangente à chaque circonférence. Trouver le point pour lequel la somme des deux tangentes ainsi obtenues est aussi petite que possible, et celui pour lequel leur différence est aussi grande que possible.

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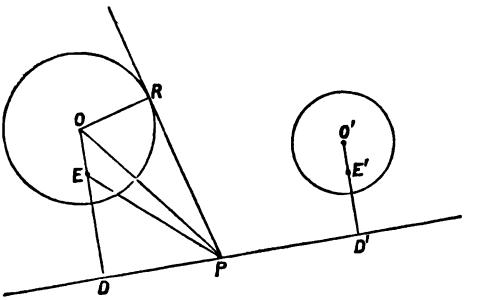
Solution by R. F. Davis, M.A.



From the centres O, O′ let fall the perpendiculars OD, OD′ upon the given straight line. Upon OD take a point E, such that

DE^2 = square of tangent from D to the first circle = $OD^2 - r_1^2$.

Then, if P be any point on DD′, the square of the tangent from P to the first circle = $OP^2 - OR^2 = OD^2 + PD^2 - r_1^2 = DE^2 + PD^2 = PE^2$.

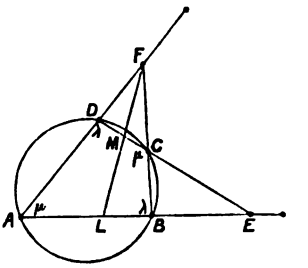
Let E′ be the corresponding point on O′D′. Then the problem reduces to:—Given E, E′ fixed points, to find a point P upon a fixed straight line DD′, such that EP + E′P is a minimum, or EP~E′P a maximum. The solutions of this are elementary and well known.

11218. (Professor Genese, M.A.) — The bisectors of the angles between opposite sides of a quadrilateral inscribed in a circle meet on the line joining the middle points of the diagonals, and divide that line internally and externally in the ratio of the diagonals.

Solution by R. F. Davis, M.A.

Let ABCD be a quadrilateral inscribed in a circle; and let its opposite sides AB, DC and AD, BC intersect in E, F, respectively. Then, by similar triangles,

$$EA : ED = FC : EB = FA : FB$$
$$= FC : FD = AC : BD$$
$$= \lambda : \mu, \text{ say.}$$



Place masses μ, λ, μ, λ, at A, B, C, D, respectively.

Then the centroid of μ at A and λ at B is at a point L in AB, such that

$$AL : LB = \lambda : \mu.$$

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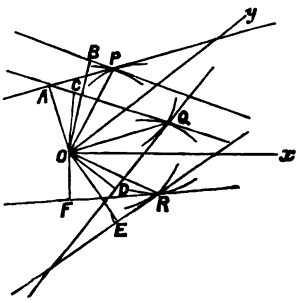
But FA : FB = $\lambda : \mu$; hence FL bisects the angle AFB. Similarly, the centroid M of μ at C and λ at D lies on the bisector of the angle CFD; so that F, M, L are collinear. Thus the centroid S of the four masses coincides with the point of intersection of the lines bisecting the angles E, F.

But, by taking the masses in opposite pairs, it is easily seen that S lies on the line joining the middle points of AC, BD, and dividing it in ratio $\lambda : \mu$. [See, in MILNE'S *Companion*, Mr. DAVIS'S chapter on the Geometrical Illustrations of Centroidal Properties.]

6490. (Professor HUDSON, M.A.)—A series of lemniscates are drawn having the same axis and node, and another series with the same node and different axis; prove that any lemniscate of the first set cuts any of the second at an angle which is twice the angle between the axes of the two sets.

Solution by Professor G. B. M. ZERR.

Let OX, OY be the initial lines; $r^m = a^m \cos m\theta$ represent the series of curves with OX as axis; $s^m = b^m \cos m\phi$ represent the series of curves with OY as axis.



Then there are three cases:—

(1) Let P be the point of intersection, OA, OB the perpendiculars from origin on tangents; then

AOB = AOP − POB = $m\theta - m\phi$

= angle of intersection.

But $\phi = \theta - \beta$, where β is the angle between the axes;

$\therefore\ m\theta - m\phi = m\theta - m(\theta - \beta) = m\beta.$

(2) Q is the point of intersection, and OC, OD the perpendiculars from origin on tangents;

COD = $m\theta + m\phi$ = angle of intersection. But $\phi = \beta - \theta$;

$\therefore\ m\theta + m\phi = m\theta + m(\beta - \theta) = m\beta.$

(3) R is the point of intersection, and OE, OF the perpendiculars from origin on tangents,

EOF = ROF − ROE = $m\phi - m\theta$ = angle of intersection. But $\phi = \beta + \theta$;

$\therefore\ m\phi - m\theta = m(\beta + \theta) - m\theta = m\beta.$

Let $m = 2$, and the curves are lemniscates, and then $m\beta = 2\beta$.

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11383. (Professor STEGGALL.)—There are twelve electors and three candidates A, B, C for an office. On being put up together, A gets more votes than B, B than C. On being put up in pairs, A is preferred to B, B to C, and C to A; what were the original votes?

Solution by H. W. CURJEL, B.A.

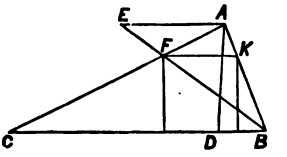
At least 5 vote for A originally, and not more than 5 vote for A when A and C are put up together; therefore originally 5 vote for A; therefore 4 for B, and 3 for C.

11578. (A. J. PRESSLAND, M.A.)—If k be the side of the square inscribed in a triangle ABC, having one side collinear with BC, and p be the length of the perpendicular from A on BC, prove that

$$R^{-1} = a^{-1} + p^{-1}.$$

Solution by T. W. K. CLARKE; A. KAHN, B.A.; *and others.*

To construct such a square, we draw AE equal and perpendicular to AD, and join EB cutting AC in F; then FK drawn parallel to BC is a side of the square.



Now $\frac{k}{a} = \frac{AK}{AB}$ and $\frac{k}{p} = \frac{BK}{AB}$,

since FK is parallel to BC and to EA;

therefore $$\frac{k}{a} + \frac{k}{p} = 1.$$

11552. (Professor NILKANTHA SARKAR.)—Prove that

$$\int_0^\infty \frac{x^4 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2} \frac{bc+ca+ab}{(b+c)(c+a)(a+b)}.$$

Solution by T. W. K. CLARKE; A. KAHN, B.A.; *and others.*

$$\int_0^\infty \frac{x^4 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \Sigma \int_0^\infty \frac{a^4 dx}{(x^2+a^2)(a^2-b^2)(a^2-c^2)}$$

$$= \frac{\pi}{2} \Sigma \frac{a^3}{(a^2-b^2)(a^2-c^2)} = \frac{\pi}{2} \cdot \frac{bc+ca+ab}{(b+c)(c+a)(a+b)}.$$

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11535. (H. Brocard.)—Au point P d'une courbe Δ on mène l'ordonnée PB et la tangente PA rencontrant l'axe OX en A; la perpendiculaire élevée en A sur PA coupe PB en M. Déterminer la courbe Δ par la condition que la courbe (M) soit symétrique de Δ par rapport à l'origine.

Solution by H. W. Curjel, B.A.; Professor Bhattacharya; *and others.*

Let $y = f(x)$ be the equation of the curve Δ; x, y the coordinates of P; and x, η those of M; then $(-x, -\eta)$ is on Δ.

$$\eta y = -AB^2 = -\frac{y^2}{\left(\frac{dy}{dx}\right)^2}; \qquad \therefore\ f(-x) = -\eta = \frac{y}{\left(\frac{dy}{dx}\right)^2};$$

hence
$$[f'(x)]^2 = \frac{f(x)}{f(-x)} \qquad \text{.....(1);}$$

$$+[f'(-x)]^2 = \frac{f(-x)}{f(x)}; \qquad [f'(-x)]^2 = \frac{1}{[f'(x)]^2} \qquad \text{.....(2).}$$

Differentiating (1), $\frac{2f''(x)}{f'(x)} = \frac{f'(x)}{f(x)} + \frac{f'(-x)}{f(-x)} = \frac{2f'(x)}{f(x)}$ or 0,

the latter gives $y'' = 0$, or $y = ax + b$, which obviously does not satisfy the conditions; therefore

$$\frac{y''}{y'} = \frac{y'}{y}; \qquad \therefore\ y' = ky; \qquad \therefore\ y = ae^{x/c},$$

where a and c are arbitrary constants; but

$$(y')^2 = \frac{ae^{x/c}}{ae^{-x/c}} = e^{2x/c}; \qquad \therefore\ a = c;$$

therefore equation to Δ is $y = ae^{x/a}$.

11520. (Professor Neuberg.)—Trouver

$$\int \frac{\sin 3x}{\cos 4x}\,dx.$$

Solution by Vincent J. Bouton, B.Sc.; Prof. Paroling; *and others.*

$$\int \frac{\sin 3x}{\sin 4x}\,dx = \tfrac{1}{4}\int \frac{(3-4y^2)\,dy}{(1-y^2)(1-2y^2)}, \quad \text{where } y \equiv \sin x,$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}\int \frac{dy}{1+y} + \tfrac{1}{2}\int \frac{dy}{1-y} + \int \frac{dy}{1+\sqrt{2}\,y} + \int \frac{dy}{1-\sqrt{2}\,y}\right)$$

$$= \tfrac{1}{8}\log\frac{1+\sin x}{1-\sin x} + \frac{1}{4\sqrt{2}}\log\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}.$$

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11572. (Morgan Brierley.)—P is a variable point external to a circle given in magnitude and position, and PD a line making a given angle with a diameter of the circle produced; find the *locus* of P, when $PD^2 + PC^2$ is a given magnitude, PC being a tangent to the circle.

Solution by A. Kahn, B.A.; R. Knowles, B.A.; *and others.*

We have $PD^2 + PC^2 = K^2$, where K is a constant; hence, if the diameter be taken as axis of x, and a diameter perpendicular to it as axis of y,

$$y^2 \operatorname{cosec}^2 \alpha + y^2 + x^2 - a^2 = K^2,$$

where x, y are the coordinates of P, a the radius, and α the given angle. Thus the locus of P is the ellipse

$$\frac{x^2}{K^2 + a^2} + y^2 \Big/ \frac{(K^2 + a^2)\sin^2\alpha}{1 + \sin^2\alpha} = 1.$$

11276. (D. Edwardes, B.A.)—Given that

$$x/(m^2\alpha^2 - \beta\gamma)^{\frac{1}{2}} = y/(m^2\beta^2 - \gamma\alpha)^{\frac{1}{2}} = z/(m^2\gamma^2 - \alpha\beta)^{\frac{1}{2}},$$

$$m^2(\alpha^3 + \beta^3 + \gamma^3) = (1 + 2m^3)\,\alpha\beta\gamma,$$

prove that

$$\alpha/(m^2x^2 - yz)^{\frac{1}{2}} = \beta/(m^2y^2 - zx)^{\frac{1}{2}} = \gamma/(m^2z^2 - xy)^{\frac{1}{2}},$$

$$m^2(x^3 + y^3 + z^3) = (1 + 2m^3)\,xyz.$$

Solution by H. W. Curjel, B.A.

$$\frac{x^2}{m^2\alpha^2 - \beta\gamma} = \frac{y^2}{m^2\beta^2 - \gamma\alpha} = \frac{z^2}{m^2\gamma^2 - \alpha\beta} = \frac{yz}{(m^4\beta^2\gamma^2 + m^2\alpha^4 - 2\alpha\,\beta\gamma m^3)^{\frac{1}{2}}}$$

$$= \frac{yz}{m\alpha^2 - m^2\beta\gamma} = \frac{zx}{m\beta^2 - m^2\gamma\alpha} = \frac{xy}{m\gamma^2 - m^2\alpha\beta}$$

$$= \frac{m^2x^2 - yz}{\alpha^2(m^4 - m)} = \frac{m^2y^2 - xz}{\beta^2(m^4 - m)} = \frac{m^2z^2 - xy}{\gamma^2(m^4 - m)} = \frac{mx^2 - m^2yz}{(m^4 - m)\,\beta\gamma};$$

therefore

$$\frac{\alpha}{(m^2x^2 - yz)^{\frac{1}{2}}} = \frac{\beta}{(m^2y^2 - zx)^{\frac{1}{2}}} = \frac{\gamma}{(m^2z^2 - xy)^{\frac{1}{2}}},$$

$$(mx^2 - m^2yz)^2 = (m^2y^2 - xz)(m^2z^2 - xy);$$

therefore

$$m^2x^4 - 2m^3x^2yz + m^4y^2z^2 = m^4y^2z^2 - m^2x(z^3 + y^3) + x^2yz;$$

therefore

$$m^2(x^3 + y^3 + z^3) = (1 + 2m^3)\,xyz.$$

11485. (Professor Zerr.)—1000 c.c. of hydrogen precipitates 9·65 grams of silver from a solution of silver acetate; also, by heating platinum tetrachloride in 1000 c.c. of hydrogen, and precipitating the hydrogen

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chloride with the silver nitrate formed from the silver above found, I have 4·4061 grams of platinum left, when 12·8263 grams of silver chloride are formed. What is the atomic weight of silver and platinum?

Solution by H. J. Woodall; Professor Bhattacharya; *and others.*

$$4\,Ag(C_2H_3O_2) + 2H_2 = 2Ag_2 + 4C_2H_4O_2,$$
$$2\,Ag_2 + (HNO_3) = 4\,AgNO_3 + \&c., \quad PtCl_4 + 2H_2 = 4HCl + Pt,$$
$$4\,(HCl + AgNO_3 = AgCl + HNO_3),$$

4·4061 Pt equivalent to 12·8263 (4 AgCl) and to 9·65 ($2\,Ag_2$), and to 1000 c.c. $2H_2$ = 0·0896 grams; therefore atomic weight of Ag is 9·65/0·0896 = 107·7; of Pt is 4 × 4·4061/0·0896 = 196·7; and of Cl is 3·1763/0·0896 = 35 45.

2056. (The Editor.)—Prove that the straight lines joining the in- and circum-centres will subtend a right angle at the orthocentre if

$$1 + (1 - 2\cos A)(1 - 2\cos B)(1 - \cos C) = 8\cos A\cos B\cos C.$$

Solution by Professor G. B. M. Zerr.

Let H, O, I be the ortho-, circum-, and in-centres of the triangle; then (see Feuerbach's proof of his theorem on p. 185 of Milne's *Companion*)

$$(HI)^2 = 2r^2 - 4R^2\cos A\cos B\cos C, \quad (HO)^2 = R^2 - 8R^2\cos A\cos B\cos C,$$
$$(OI)^2 = R^2 - 2Rr.$$

When $\angle OHI$ is a right angle, we have

$$(HI)^2 + (HO)^2 = (OI)^2, \quad r^2 + Rr = 6R^2\cos A\cos B\cos C.$$

But $r = a\sin\frac{1}{2}B\sin\frac{1}{2}C/\cos\frac{1}{2}A$, $R = a/4\sin\frac{1}{2}A\cos\frac{1}{2}A$;

$\therefore\ 8\sin^2\frac{1}{2}A\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C + 2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C = 3\cos A\cos B\cos C.$

But $1 - \cos A = 2\sin^2\frac{1}{2}A$, $1 - \cos B = 2\sin^2\frac{1}{2}B$,
$1 - \cos C = 2\sin^2\frac{1}{2}C$,

$$\cos A + \cos B + \cos C + 1 = 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

Hence $2(1 - \cos A)(1 - \cos B)(1 - \cos C) + \cos A + \cos B + \cos C - 1$
$= 6\cos A\cos B\cos C,$

or $1 - \cos A - \cos B - \cos C + 2\cos A\cos B + 2\cos A\cos C + 2\cos B\cos C$
$= 8\cos A\cos B\cos C.$

Multiply the last equation throughout by 2, and we get

$$2 - 2\cos A - 2\cos B - 2\cos C + 4\cos A\cos B + 4\cos A\cos C + 4\cos B\cos C$$
$$= 16\cos A\cos B\cos C,$$

or $1 + (1 - 2\cos A)(1 - 2\cos B)(1 - 2\cos C) = 8\cos A\cos B\cos C.$

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11394. (H. W. Segar, B.A.)—There is a series of integers such that, if they be denoted by $u_1, u_2, u_3 \ldots$, and their successive differences by $v_1, v_2, v_3, \ldots$, then the following relations hold:—

$$u_{2n} = u_n^2 + v_n^2, \quad u_{2n+1} = u_{n+1}^2 + v_n^2, \quad v_{2n} = u_{n+1}^2 - u_n^2, \quad v_{2n+1} = v_{n+1}^2 - v_n^2.$$

Solution by H. W. Curjel, B.A.

The last two relations may be deduced from the first two, so that if there are two integers which satisfy the equation $u_2 = u_1^2 + (u_2 - u_1)^2$, taking them as the first two terms, the rest of the series may be deduced in succession and will be integral. The first few terms 1, 2, 5, 13, 34, 89, 233, 610, satisfy the equation $u_{n+1} - 3u_n + u_{n-1} = 0$; also

$$u_{2n+1} - 3u_{2n} + u_{2n-1} = (u_{n+1} - 3u_n + u_{n-1})(-u_{n+1} + u_n + u_{n-1}),$$

and $$u_{2n+2} - 3u_{2n+1} + u_{2n} = (u_{n+2} - 3u_{n+1} + u_n)(u_{n+2} - 3u_{n+1} + u_{n-1});$$

hence all the terms satisfy the equation $u_{n+1} - 3u_n + u_{n-1} = 0$. Hence

$$u_n = A\left(\frac{3-5^{\frac{1}{2}}}{2}\right)^n + B\left(\frac{3+5^{\frac{1}{2}}}{2}\right)^n$$

$$= \frac{2}{5^{\frac{1}{2}}(5^{\frac{1}{2}}-1)}\left(\frac{3-5^{\frac{1}{2}}}{2}\right)^n + \frac{2}{5^{\frac{1}{2}}(5^{\frac{1}{2}}+1)}\left(\frac{5^{\frac{1}{2}}+3}{2}\right)^n \text{ (for } u_1 = 1 \text{ and } u_2 = 2)$$

$$= \frac{1}{5^{\frac{1}{2}}}\left\{\left(\frac{5^{\frac{1}{2}}-1}{2}\right)^{2n-1} + \left(\frac{5^{\frac{1}{2}}+1}{2}\right)^{2n-1}\right\}.$$

11448. (Professor Catalan.)—On prend p nombres, dans la suite 1, 2, ... n. On fait la somme de ces p nombres. Evaluer la somme de toutes ces sommes partielles.

Solution by J. H. Grace; H. W. Curjel, B.A.; *and others.*

There will be $\frac{n!}{p!\,(n-p)!}$ partial sums, and the numbers will evidently all occur the same numbers of times. Therefore the number of times each number occurs $= \frac{n!}{p!\,(n-p)!} \times \frac{p}{n} = \frac{(n-1)!}{(p-1)!\,(n-p)!}$;

therefore the sum of the partial sums

$$= \frac{(n-1)!}{(p-1)!\,(n-p)!} \times \frac{n(n+1)}{2} = \frac{(n+1)!}{2(p-1)!\,(n-p)!}.$$

11435. (G. J. Griffiths, M.A.)—Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime is $15\pi^{-2}$.

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Solution by H. W. CURJEL, B.A.

$$\text{Required sum} = \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots$$

$$= \frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots}.$$

But
$$1 \Big/ \left\{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\right\}$$

$$= \left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\ldots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\right)\ldots$$

$$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots = \tfrac{1}{6}\pi^2,$$

$$1 \Big/ \left\{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\ldots\right\}$$

$$= \left(1+\frac{1}{2^4}+\frac{1}{2^8}+\frac{1}{2^{12}}+\ldots\right)\left(1+\frac{1}{3^4}+\frac{1}{3^8}+\frac{1}{3^{12}}+\ldots\right)\ldots$$

$$= \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\ldots = \tfrac{1}{90}\pi^4.$$

Therefore the required sum $= 15\pi^{-2}$.

9296. (A. KAHN, B.A.)—Find the developable whose edge of regression is the curve of intersection of the two surfaces

$$x^2-y^2-4z^2 = 0, \quad yz+zx-a(x-y) = 0 \ldots\ldots\ldots\ldots\ldots (1, 2).$$

Solution by H. J. WOODALL.

From the given equations, we get

$$\pm x = (az)^{\frac{1}{2}} + (z^3/a)^{\frac{1}{2}}, \quad \pm y = (az)^{\frac{1}{2}} - (z^3/a)^{\frac{1}{2}} \ldots\ldots\ldots\ldots (3, 4).$$

Intersection of tangent planes to (3) and (4),

$$x-x' = \tfrac{1}{2}\left\{(a/z')^{\frac{1}{2}} + 3(z'/a)^{\frac{1}{2}}\right\}(z-z'),$$

$$y-y' = \tfrac{1}{2}\left\{(a/z')^{\frac{1}{2}} - 3(z'/a)^{\frac{1}{2}}\right\}(z-z'),$$

$$x' = (az')^{\frac{1}{2}} + (z'^3/a)^{\frac{1}{2}} = (az')^{\frac{1}{2}}(1+z'/a), \quad y' = (az')^{\frac{1}{2}}(1-z'/a).$$

Eliminating x', y', z', we have

$$x-(az')^{\frac{1}{2}}(1+z/a) = \tfrac{1}{2}(a/z')^{\frac{1}{2}}(1+3z'/a)(z-z'),$$

$$y-(az')^{\frac{1}{2}}(1-z'/a) = \tfrac{1}{2}(a/z')^{\frac{1}{2}}(1-3z'/a)(z-z').$$

Eliminating z', we get finally

$$3x^2(x^2-y^2) = \left\{a(x-y)(4x+2y)+9xz(x+y)-3ax(x-y)\right\}^2/9(x+y)^2,$$

the developable.

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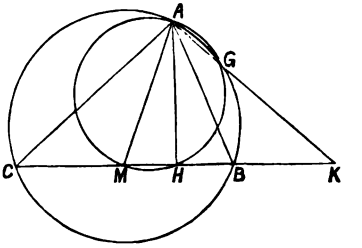
11414. (Professor MANNHEIM.)—Sur la médiane *am* d'un triangle *abc*, comme diamètre, on décrit une circonférence de cercle. Cette courbe coupe, au point *g*, le cercle circonscrit au triangle *abc*; démontrer que les droites *ab*, *ac*, *ag*, et la hauteur *ah* du triangle, forment un faisceau harmonique.

Solution by T. W. K. CLARKE; Professor RADIKANSHUAN; *and others.*



Let AG meet BC in K. Now H is on the circle upon AM as diameter. Therefore

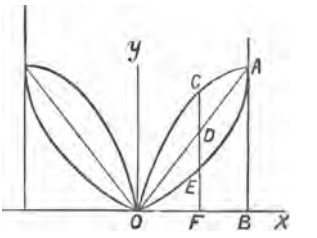
$$KB.KC = KG.KA = KH.KM$$
$$= \tfrac{1}{2}KH(KB+KC);$$

hence $$\frac{2}{KH} = \frac{1}{KC} + \frac{1}{KB},$$

KBHC is an harmonic range, and AB, AC, AG, AH form an harmonic pencil.

2140. (The EDITOR.) — Trace the curve whose equation is $x = 2a \sin \frac{y}{x}$; and prove that each loop has the same area πa^2, and is bisected by the straight line that joins the origin to the point where the tangent is parallel to the axis of y.

Solution by Professor G. B. M. ZERR.



Writing the equation in the form $y = x \sin^{-1}(x/2a)$, we see that the curve is possible for all values of x from $-2a$ to $+2a$, and y is always positive. The figure is somewhat as the diagram. Area of each loop

$$= \int_0^{2a} x \sin^{-1} \frac{x}{2a}\, dx = \pi a^2.$$

When $x = 2a$, $dy/dx = \infty$. Therefore AB is tangent to the curve at A, and the curve is symmetrical with respect to the line OA, or $y = \frac{1}{2}\pi x$. Therefore DC = DE, and OA bisects the loop. Also the curve AEO bisects the right-triangle ABO.

11575. (J. C. ST. CLAIR.)—If P be the point of concurrence of the lines joining the vertices of a triangle ABC to the points of contact of any inscribed conic with the opposite sides, prove that the tangents at A, B, C to any conic passing through A, B, C, P form a triangle which is self-conjugate with respect to the first conic.

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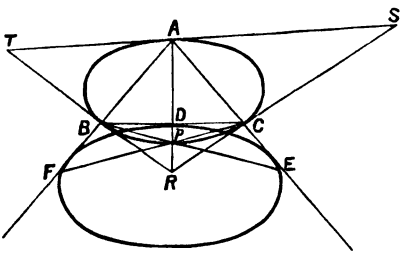
Solution by H. W. CURJEL, B.A.; J. BURKE, M.A.; *and others.*

Let D, E, F be the points of contact of the inscribed conic with the sides BC, CA, AB of the triangle; then D is the pole of FE with respect to conic ABCP, since ABCP is a quadrilateral inscribed in the conic ABCP, and D is the intersection of two diagonals of ABCP, and FE is the other diagonal.

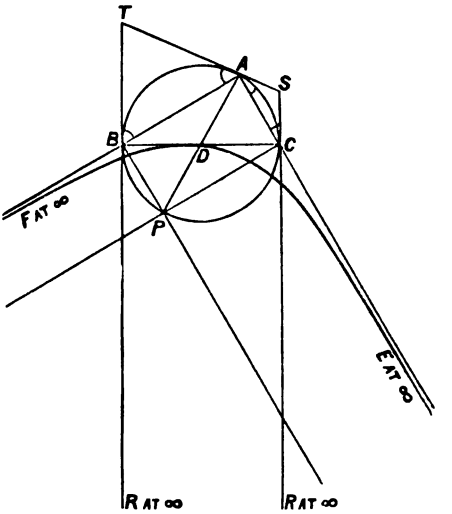


Hence, if we project the points in which FE cuts conic ABCP into the circules, conic ABCP and D projects into a circle and its centre D, and EF into the line at infinity.

Therefore, in projected figure, $\angle BAC$ becomes a right angle, and the circumscribed conic becomes a rectangular hyperbola with asymptotes AB, AC, &c.

Since tangents at B and C to the circle are parallel, and make angles with AB, AC equal to those which tangent at A makes with AB, AC, intersection R of tangents at B and C (at infinity) is the pole of tangent at A with respect to the hyperbola.

Hence, in the original figure the tangent at A is the pole of the intersection of the tangents at B and C, with respect to the inscribed conic of triangle ABC.



Similarly, we may show that the tangent at B is the polar of the intersections of tangents at A and C, and similarly for tangent at C.

Therefore triangle of tangents at A, B, C is self-conjugate with respect to conic DEF.

10208. (J. J. BARNIVILLE, M.A.)—Inscribe in a quadrant a square having two angles on the circumference.

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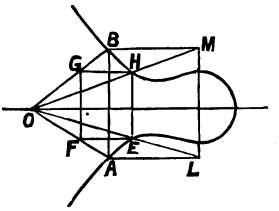
Solution by D. BIDDLE.

The following is a general construction for the case of any curve symmetrical about an axis bisecting the vertical angle, the angular capacity of the sector being arbitrary. On AB draw the square ALMB; then the points E, H, where OL, OM meet the curve, are two of the angles required. It is obvious that if OL, OM, met the symmetrical branches in more than one point each, we should have several constructions. [This method is more general than Mr. GALLIERS' solution, given on p. 92 of Vol. LII.]



11559. (Professor BERNÈS.)—Dans l'angle A d'un triangle ABC et dans son opposé par le sommet, on trace deux droites DE, D'E' antiparallèles à BC et équidistantes de A. Faire voir que, si M est la rencontre de BE et CD, et M' la rencontre de BE', CD', les droites AM, AM' sont conjuguées harmoniques relativement à la médiane et à la symédiane issues de A dans le triangle ABC.

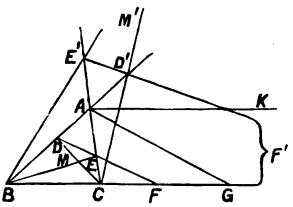
Solution by W. J. GREENSTREET, M.A.; Professor SARKAR; *and others.*

Draw AG, AK parallel to DE, BC, respectively.

Then AG is the tangent at A to the circum-circle of ABC, and GC/GB = b^2/c^2. The polar of G with respect to BAC will cut BC in G', where

$$G'CG'B = GC/GB = b^2/c^2.$$

We also know the straight line from A, symmetric to the median with respect to the bisector of A, *i.e.* the symmedian, cuts BC in the ratio of $b^2 : c^2$.



Hence A(KF'GF) is an harmonic pencil (because DE, D'E' are equidistant from A, and G bisects FF'); the polar of AK is the A-median; the polar of AF' is AM'; the polar of AG is the A-symmedian; and the polar of AF is AM.

Now if four concurrent lines form an harmonic pencil, so do their polars. Therefore, &c.

11548. (Professor HUDSON, M.A.)—From two points in a plane two series of straight lines are drawn at the same equal small angular intervals, dividing the plane into quadrilateral elements; find (1) the angle

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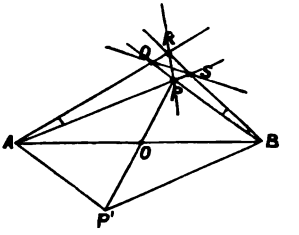
between the diagonals of these elements ultimately when the angular intervals are indefinitely small, and (2) the locus of points such that this angle is a right angle.

Solution by Professor SCHOUTE.

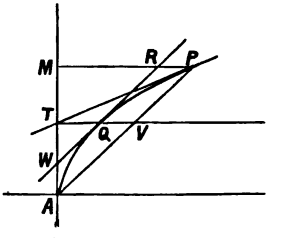
Let A, B be the two given points, and P any point of the plane. If the equal angles PAR and PBR diminish, the quadrangle PQRS is a parallelogram similar to the parallelogram APBP′, for these figures have a common angle, and the adjacent sides are proportional. This proves (1) that the required angle is equal to the angle POB, and (2) that the locus of P, for which the required angle is right, is the normal in O on AB.



9404. (Rev. T. C. SIMMONS, M.A.)—If the tangent at any point P of a parabola meet the tangent at the vertex A in T, and the diameter TQV meet the curve in Q and AP in V; then, drawing PM at right angles to AT, and the tangent at Q to meet PM in R and AM in W, the ratios of (1) PV to PA, (2) TQ to TV, (3) triangle RMW to triangle PMA, (4) curvilinear area AQPV to triangle ATP, are all constant. Prove that these properties hold for all parabolic curves ($y = cx^m$), and that the respective ratios are denoted by (1) m, (2) $(1-m^{-1})^{m-1}$, (3) $(1-m^{-1})^{m-1}$, (4) $m/(m^2-1)$.

Solution by Professor G. B. M. ZERR.

Let AQP be a portion of one of the parabolic curves $y = cx^m$, PT any tangent at the point P (x_1, y_1).



Then the equation to PT is

$$y_1 - y = MCx_1^{m-1}(x_1 - x);$$

$y = (1-m)y_1$ is equation to TV;

$\{(1-m)x_1, (1-m)y_1\}$ are coordinates of V;

(1) $\therefore$ PV : PA :: $M(x_1^2+y_1^2)^{\frac{1}{2}} : (x_1^2+y_1^2)^{\frac{1}{2}}$

or $PV/PA = M$.

(2) $$TQ = (1-m)^{1/m}x_1, \quad TV = (1-m)x_1;$$

$$\therefore \quad TQ/TV = (1-m)^{(1-m)/m}.$$

(3) $$(1-m)y_1 - y = mc\left\{(1-m)^{1/m}x_1\right\}^{m-1}\left\{(1-m)^{1/m}x_1 - x\right\}$$

is equation to WR.

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Hence $\quad MR = \dfrac{(2-m)\,x_1}{(1-m)^{(m-1)/m}}, \quad MW = M\,(2-m)\,y_1.$

$$\text{Area RMW} = \frac{m\,(2-m)^2\,x_1 y_1}{2\,(1-m)^{(m-1)/m}}, \quad \text{area PMA} = \tfrac{1}{2}x_1 y_1.$$

$$RMW/PMA = m\,(2-m)^2\,(1-m)^{(1-m)/m}.$$

(4) $\quad \text{Area AQPV} = \dfrac{(1-m)\,x_1 y_1}{2\,(1+m)}, \quad \text{area ATP} = \tfrac{1}{2}\,(1-m)\,x_1 y_1;$

$$\therefore \quad AQPV/ATP = (1+m)^{-1}.$$

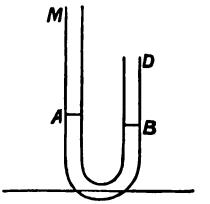
When $\quad M = \tfrac{1}{2}$, we get the parabola $y^2 = c^2 x$,

and $\quad (1) = \tfrac{1}{2}, \quad (2) = \tfrac{1}{2}, \quad (3) = \tfrac{9}{16}, \quad (4) = \tfrac{8}{9}.$

5301. (Rev. E. Hill, M.A.)—Certain persons have imagined the existence of a subterranean connexion between the waters of the Dead Sea and the Mediterranean. Although the difference of their levels is 1300 feet, yet, since the ratio of their densities is 1·24, it is possible that such a passage may exist. But find its necessary depth.

Solution by H. J. Woodall.

For simplicity, suppose the tube be in shape of a U.



Distances measured above the datum line drawn, M is x ft. high, A is y ft. high, D is $(x-1300)$ ft. high, B is z ft. high.

Then, supposing that another fluid, sp. gr. $= g$, fills the bend, we have, for equilibrium,

$$(x-y)\times 1 + yg = (x-1300-z)\,1{\cdot}24 + zg,$$

$$\therefore \quad x\times{\cdot}24 + y = (1300+z)\,1{\cdot}24 + (y-z)\,g,$$

$$\text{or } x\times{\cdot}24 = 1300\times 1{\cdot}24 + y\,(g-1) + z\,(1{\cdot}24-g).$$

In the case given, $\quad x\times{\cdot}24 = 1300\times 1{\cdot}24$

gives $\quad x = 6716\tfrac{2}{3}$ ft. $= 1\tfrac{5}{16}$ miles about.

This is supposing that the density does not vary, in other cases a different form would need to be given to the methods of solution.

11356. (E. M. Langley, M.A.)—ABC is an equilateral triangle, D a point in BC such that BD is one-third of BC, and E is a point in AB equidistant from A and D. Show geometrically that $CE = EB + BD$.

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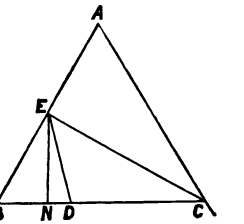
Solution by JAMES RICE; Professor BHATTACHARYA; *and others.*



Draw EN perpendicular to BC; let $3a$ = the side of the equilateral triangle, and let EA = ED = b; then

EB = $3a-b$, $2DN = 2BD-2BN = b-a$,

$EB^2-ED^2 = BN^2-DN^2$,

$3a(3a-2b) = a\left[\frac{1}{2}(3a-b)-DN\right]$,

$2DN = 11b-15a$, $11b-15a = b-a$;

$\therefore\ b = \frac{7}{5}a$, $EB+BD = \frac{13}{5}a$;

also $CE^2 = CD^2+ED^2+2CD\,.\,DN = \frac{169}{25}a^2$;

therefore $CE = \frac{13}{5}a = EB+BD$.

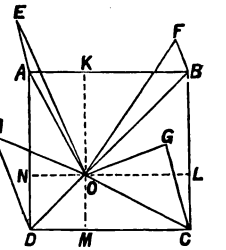
[*Otherwise*:—$EC^2 = EA^2+AC^2-EA\,.\,AC = EA^2+EB\,.\,AC$

$= ED^2+3EB\,.\,BD$;

also $ED^2 = EB^2+BD^2-EB\,.\,BD$; hence $EC^2 = EB^2+BD^2+2EB\,.\,BD$.]

11545. (Professor ZERR.)—At the corners of a square, of side d, there are four towers of heights a, b, c, $(a^2+c^2-b^2)^{\frac{1}{2}}$. Find the length of a ladder and the distance of the point from the foot of each tower, at which, if the ladder be placed, it will just reach the top of each tower.

Solution by H. J. WOODALL; L. WIENER; *and others.*



Let AE = a, BF = b, CG = c,

DH = $(a^2+c^2-b^2)^{\frac{1}{2}}$, OE = OF = OH = OG = x,

AN = BL = u, DN = CL = v,

AK = DM = y, KB = MC = z.

Then $x^2 = a^2+u^2+y^2 = b^2+u^2+z^2$

$= c^2+v^2+z^2$

$= a^2+c^2-b^2+y^2+v^2$... (1, 2, 3, 4).

(2) from (1) gives $a^2-b^2 = z^2-y^2$,

but $z+y = d$;

$\therefore\ z = \frac{a^2-b^2+d^2}{2d}$, $y = \frac{d^2+b^2-a^2}{2d}$.

Similarly, from (2) and (3), $v = \frac{b^2-c^2+d^2}{2d}$, $u = \frac{d^2+c^2-b^2}{2d}$.

But AO = $(u^2+y^2)^{\frac{1}{2}}$, BO = $(z^2+u^2)^{\frac{1}{2}}$, CO = $(v^2+z^2)^{\frac{1}{2}}$, DO = $(y^2+v^2)^{\frac{1}{2}}$, and the problem is solved.

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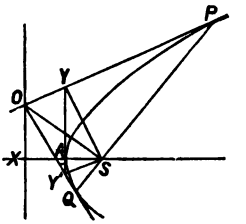
11576. (W. J. GREENSTREET, M.A.)—A parabola always touches the rectangular axes so that the chord of contact is a constant (a); show that the locus of its focus is a curve $(x^2+y^2)^3 = a^2x^2y^2$.

Solution by T. W. K. CLARKE; A. KAHN, B.A.; *and others.*



Let PQ be the chord of contact; then S, the foot of the perpendicular from O on PQ, is the focus; now we have

$$OS = a\cos\theta\sin\theta,$$

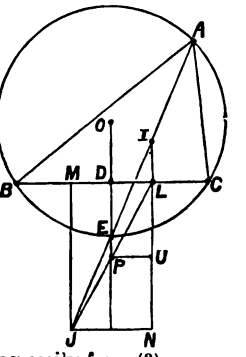
hence the locus of the focus is

$$r = a\cos\theta\sin\theta,$$

or $$(x^2+y^2)^3 = a^2x^2y^2.$$

11242. (W. J. GREENSTREET, M.A.) — Draw three circles, each passing through two angular points of a triangle and cutting its incircle orthogonally. Take the triangle formed by the joins of the centres of these circles, and prove that (1) its sides are parallel to contact joins of incircle; (2) the centres of the circumcircles of both triangles are identical; and (3) the radii of the circumcircles differ by $\frac{1}{2}$ radius of the incircle.

Solution by R. F. DAVIS, M.A.



Let ABC be a triangle, O its circum-centre, D the middle point of BC, E the point in which OD produced meets the circum-circle. Then EA bisects the angle at A; and, if upon it points I, J be taken, such that EI = EJ = EB (or EC), then I, J are the centres of the in- and ex-scribed circles. Join J L, meeting OE in P, so that

$$EP = \tfrac{1}{2}\text{ radius of in-circle};$$

then $PC^2 - PL^2 = CD^2 - DL^2 = BL \cdot LC$

$= IL \cdot LN$ (since I, B, J, N, C are concyclic)

$= IL \cdot IN - IL^2 = PI^2 - PL^2 - IL^2,$

or $$PC^2 = PI^2 - IL^2.$$

Hence the circle of centre P and radius PC intersects the in-circle orthogonally.

Hence, &c., the properties (1) and (2) following easily from (3).

11432. (I. ARNOLD.)—Construct an equilateral triangle that shall be equal to a given parallelogram.

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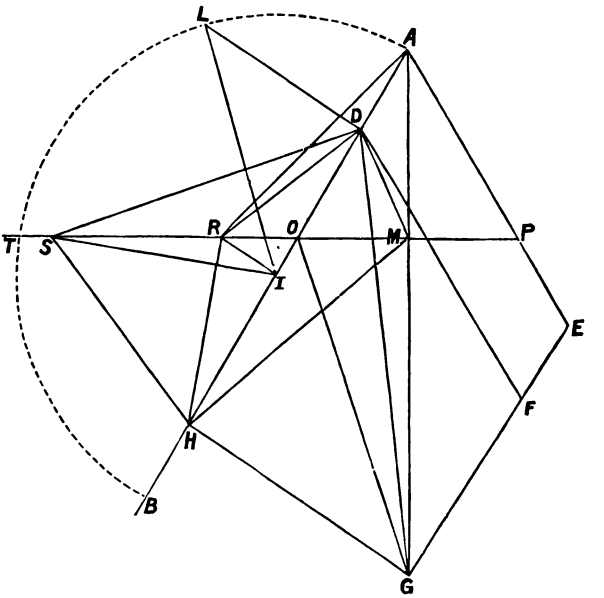
Solution by the PROPOSER.

Make the angle BAE equal the angle of an equilateral triangle. In AB take AD, and to it apply the parallelogram ADFE equal the given



parallelogram. Bisect the angle BAE by the line AG. Produce EF to meet AG in G. From G draw GH perpendicular to AB. Bisect HD in I and make HB = AD. From I as centre with IA or IB as radius describe the semicircle ALB, and from D draw the perpendicular DL. Make AM equal to DL, and through M draw PMOT perpendicular to AM; AOP is the equilateral triangle required.

From I draw IR perpendicular to AB, and meeting PT in R. Make RS = RM, and draw lines as in diagram; then

$$AM^2 + MR^2 = AR^2 = IR^2 + IA^2.$$

But $$IA^2 = IL^2 = ID^2 + DL^2;$$

therefore $$AM^2 + MR^2 = IR^2 + ID^2 + DL^2 = DL^2 + RD^2.$$

But $AM^2 = DL^2$; and therefore $MR^2 = RD^2$. Consequently

$$RD = RM = RH = RS;$$

and therefore a circle with R as centre and RD as radius will pass through the points D, M, H, S, and SM will be the diameter, and the angles SDM, SHM will be right angles. But the angles OMG, OHG of the quadrilateral figure MOHG are right angles; therefore the angle

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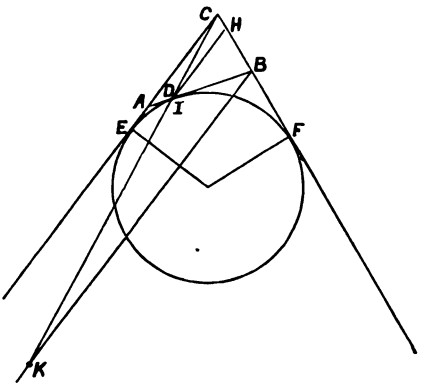
OGM = OHM, and consequently is equal to RSD or RDS; because DHM = DSM. But SDM, being a right angle, is equal to OMA, or the sum of AMD and OMD or its equal RDM; because RDM is an isosceles triangle. Take away RDM from both, and the angle RDS (and therefore its equal OGM) = DMA. Hence DM is parallel to OG. The triangle DOM is therefore equal to DGM, and if ADM be added to both, the triangles OAM and DAG are equal, and their doubles, the triangle AOP and the parallelogram ADFE. Therefore the equilateral triangle AOP is constructed equal to the given parallelogram; which was to be done.

11243. (MORGAN BRIERLEY.)—Given the vertical angle, the perimeter, and the length of a line drawn from the vertical angle to meet the base, and divide it in a given ratio, show how to determine the triangle by the properties of plane geometry without reference to conics.

Solution by the PROPOSER.



Draw two indefinite straight lines CE, CF, to contain the given vertical angle; and upon each set off CE, CF, each equal to half given perimeter; describe a circle to touch the two lines in E and F. Now find a line CK, such that CK : CD (the given line from the vertex to the base) = $m+n$: n, the given ratio of the segments of the base AB of the triangle required. With centre C, and CK as radius, draw an arc of a circle between CE, CF, in which, by a known method, determine the point K, so that the line KB, drawn parallel to CE, and meeting CF in B, may have the ratio BK : CK − CD = m : n. From B, thus determined, draw a tangent BIA to the circle which touches the sides in E, F, to meet CE in A; ABC is the triangle required. Join CK, cutting AB in D, and draw DH parallel to KB. AI = AE, and BI = BF; therefore (AI + IB =) AB + AC + CB = CE + CF = given perimeter. By similar triangles, we have

$$AB : AD = BC : CH = CK : CD = m+n : n.$$

Also CD = CK − DK, determined by construction.

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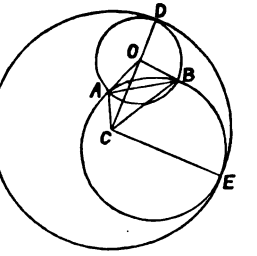
11566. (Belle Easton, B.Sc.)—If points A, B are taken within a circle of radius r whose centre is C, prove that the diameters of the circles which can be drawn through A and B to touch the given circle, are (the symbols referring to the parts of the triangle ABC) the roots of $x^2(r^2c^2 - a^2b^2\sin^2 C) - 2xrc^2(r^2 - ab\cos C) + c^2(r^4 - 2r^2ab\cos C + a^2b^2) = 0$.

Solution by Professors Zerr, Mukhopadhyay, *and others.*



Let $AO = OD = OB = \frac{1}{2}x$;

then $\cos ACO = (r^2 - rx + b^2)/(2rb - bx)$,

$$\sin ACO = \frac{\{(2rb - bx)^2 - (r^2 - rx + b^2)^2\}^{\frac{1}{2}}}{2rb - bx}.$$

Writing a for b in the above equations, we get cos BCO and sin BCO. Substituting these values in the expression

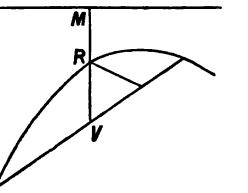
$$\cos C = \cos(ACO + BCO),$$

we get $a^2b^2(2r-x)^2\cos^2 C - 2ab\cos C(r^2 - rx + b^2)(r^2 - rx + a^2)$
$= a^2b^2(2r-x)^2 - b^2(r^2 - rx + a^2)^2 - a^2(r^2 - rx + b^2)^2$;

expanding, collecting terms, and writing $c^2 = a^2 + b^2 - 2ab\cos C$, we obtain the result stated.

5300. (R. Tucker, M.A.)—Particles are projected with a constant velocity from the same point; through the focus a straight line is drawn parallel to the direction of projection, cutting the path in P, Q. Prove that the locus of P, Q is a sphere with radius equal to twice focal distance of point of projection, and tangential to the directrix plane.

Solution by H. W. Curjel, B.A.



Let R be the point of projection, and RM the perpendicular from R on the directrix plane, which is fixed, since the velocity of projection is constant. Let QRP be one of the paths, and S its focus. Let MR meet PSQ in V. Then, since PQ is parallel to tangent at R, PQ is bisected at V, and $PV = VQ = 2RS = 2RM =$ a constant $= MV$; therefore V is fixed; therefore locus of P and Q is a sphere with V as centre, touching the directrix plane in M.

11551. (Professor Bhattacharya.)—From an unknown number of balls each equally likely to be red, white, or blue, ten are drawn, and turn

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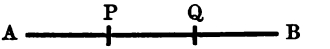
out to be five red, three white, and two blue; prove that if three more balls be drawn the chance of their being one red, one white, and one blue is $\frac{72}{455}$; the chance of three red is $\frac{8}{65}$; of three white is $\frac{4}{91}$; and of three blue is $\frac{3}{91}$; and the chance of there being no white ball in the three is $\frac{33}{91}$.

Solution by Professor ZERR; Professor RATNAM AIYAR; *and others.*

Suppose the balls arranged along the line AB, the red on AP, the white on PQ, and the blue on QB.



Let AB = 1, AQ = y, AP = x, PQ = $y-x=u$, QB = $1-y=v$; then

$$p_1 = 6\int_0^1\int_0^y x^6u^4v^3\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{72}{455};$$

$$p_2 = \int_0^1\int_0^y x^5u^3v^2\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{8}{65};$$

$$p_3 = \int_0^1\int_0^y x^5u^6v^2\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{4}{91};$$

$$p_4 = \int_0^1\int_0^y x^5u^3v^5\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{2}{91}.$$

The chance that two are red and one blue is

$$p_6 = 3\int_0^1\int_0^y x^7u^3v^3\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{63}{455}.$$

The chance that one is red and two blue is

$$p_7 = 3\int_0^1\int_0^y x^6u^3v^4\,dy\,dx \Big/ \int_0^1\int_0^y x^5u^3v^2\,dy\,dx = \frac{36}{455};$$

$$p_5 = p_2+p_4+p_6+p_7 = 33/91.$$

2061. (The EDITOR.)—Prove that the envelope of the radical axis of two circles of curvature of the ellipse $a^2y^2+b^2x^2=a^2b^2$, drawn at the ends of conjugate diameters, is the sextic (of class 6)

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left\{2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-3\right\} = \left\{3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-4\right\}^2,$$ having

asymptotes $$2\left(\frac{x}{a}\pm\frac{y}{b}\right)\pm 3 = 0;$$

four cusps, $x^2=2a^2$, $y=0$; $y^2=2b^2$, $x=0$; four acnodes, $\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{2}{3}$; and two crunodes at infinity.

Solution by Professor G. B. M. ZERR.

Let ϕ be the eccentric angle for the extremity of one diameter, then $90^\circ+\phi$ is the eccentric angle for the corresponding extremity of the

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other diameter; also let

$$(x-a_1)^2+(y-b_1)^2=r_1^2, \quad (x-a_2)^2+(y-b_2)^2=r_2^2$$

be the equations to the two circles.

Then $2x(a_1-a_2)+2y(b_1-b_2)=a_1^2-a_2^2+b_1^2-b_2^2+r_2^2-r_1^2$

is the equation to their radical axis; but

$$a_1=\frac{(a^2-b^2)}{a}\cos^3\phi, \qquad a_2=-\frac{(a^2-b^2)}{a}\sin^3\phi, \qquad b_1=-\frac{(a^2-b^2}{b}\sin^3\phi,$$

$$b_2=-\frac{(a^2-b^2)}{b}\cos^3\phi, \quad r_1=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}}{ab}, \quad r_2=\frac{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{3}{2}}}{ab};$$

$$\therefore \quad 2b(1-\sin\phi\cos\phi)(\cos\phi+\sin\phi)x+2a(1+\sin\phi\cos\phi)(\cos\phi-\sin\phi)y = 3ab\cos 2\phi$$

is the equation to the radical axis.

Eliminating ϕ from the above equation and its first differential equation, the results follow.

Substituting z for $\frac{x}{a}$, and $mz+c$ for $\frac{y}{b}$, in the equation to the envelope,

we get $m=\pm 1$, $c=\pm\frac{3}{\sqrt{2}}$;

therefore $\sqrt{2}\left(\frac{x}{a}\pm\frac{y}{b}\right)\pm 3=0$, are asymptotes.

11603. (Rev T. R. Terry, M.A.)—Three persons, whose skills are in the ratio $a:b:c$, play a match, the winner of which is to be the one who first wins two consecutive games. The winner of any game plays the remaining player, the first pair being chosen by lot. Show that the chance that the person who does not play in the first game wins the match is $\frac{1}{3}\frac{1-2x}{1-x}$, where $x(b+c)(c+a)(a+b)=abc$.

Solution by H. W. Curjel, B.A.; Professor Sitaramayer; *and others.*

Consider the case when C draws a bye (A, B, C being the players). Let chance that A wins the first game, and C wins the match $=u$, and chance that B wins the first game and C wins the match $=v$; then

$$u=\frac{a}{a+b}\frac{c}{a+c}\left(\frac{c}{b+c}+\frac{b}{c+b}u\right)=\frac{c}{b}x+ux.$$

$$\text{Also } v=\frac{b}{a+b}\frac{c}{b+c}\left(\frac{c}{a+c}+\frac{a}{a+c}v\right)=\frac{c}{a}x+vx;$$

$$\therefore \quad u+v=\left(\frac{c}{a}+\frac{c}{b}\right)x\Big/(1-x).$$

But chance that C draws a bye is $\frac{1}{3}$, therefore chance that player who draws a bye wins the match

$$=\frac{1}{3}\frac{x}{1-x}\Sigma\left(\frac{c}{a}+\frac{c}{b}\right)=\frac{1}{3}\frac{x}{1-x}\frac{(a+b)(b+c)(c+a)-2abc}{abc}=\frac{1}{3}\frac{1-2x}{1-x}.$$

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11442. (Professor SHIELDS, M.A.) — A lady owned and had her residence R in the centre of a round island of land, which had a circular field FF around its edge, thus enclosing a round piece of woodland W. One-half of the circumference of the round piece of woodland measures $570\frac{4}{7}$ rods more than its diameter, and the difference between the circumference of the whole island and its diameter is $3769\frac{27}{154}$ rods. She gave her daughter four circular fields D that just touch each other, and thus enclose one acre of land for her residence, which she also gave her; and gave her son a square piece of land S, containing the largest round field possible, the land lines circumscribed about the circular field in such a manner as to enclose one acre of land in each corner outside the circular field. Give the area of the circular field FF, and give dimensions of each child's tract.

Solution by Professor ZERR; Professor BHATTACHARYA; *and others.*

Let A, a = radii of island and woodland, 2R = side of square of son's tract, r = radius of one of daughter's fields; then

$$\pi a - 2a = 570{\cdot}8, \quad a = 500 \text{ rods}; \quad 2\pi A - 2A = 3769{\cdot}608, \quad A = 880 \text{ rods}.$$

$$\text{Area FF} = \pi(880^2 - 500^2) = 3277\tfrac{1}{2}\pi \text{ acres};$$

$$4r^2 - \pi r^2 = 160, \qquad \therefore \quad r = \frac{4\sqrt{10}}{\sqrt{4-\pi}} = 13{\cdot}65+ \text{ rods}.$$

$$\text{Area of daughter's tract} = 4\pi r^2 + 1 \text{ acre} = 15{\cdot}639+ \text{ acres};$$

$$4R^2 - \pi R^2 = 640, \qquad \therefore \quad R = \frac{8\sqrt{10}}{\sqrt{4-\pi}} = 27{\cdot}31 \text{ rods nearly}.$$

$$\text{Area of son's tract} = 4R^2 = 18{\cdot}639+ \text{ acres}.$$

11121. (G. HEPPEL, M.A.)—In several books on Geometrical Drawing the following construction is given for inscribing a regular polygon of n sides in a circle:—Divide a diameter into n equal parts. From the vertex of an equilateral triangle on the diameter draw a line passing through the *second* point of division. The intersection of this with the circle determines a vertex of the polygon. Show what is the reason for this construction, and examine how far it is accurate for different values of n.

Note by A. J. PRESSLAND, M.A.

There are some more details in regard to the problem, in addition to those given on p. 61 of Vol. LVI. The conjectured reference to DE VILLE is correct, and the solution occurs on p. 29 of his work, which received the Privilège du Roi et fut "achevé d'imprimer le premier jour du mois d'Aoust, 1628." At the end he says that "ce probleme est à estimer pour la facilité et iustesse plus grande que de tous les autres qui ont esté escrits pour ce sujet."

DE VILLE's method consists in joining the vertex of the equilateral triangle to the first point of division and doubling the resulting arc.

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The first reference I have as yet found where the second point is joined is in Bosse's *Traité des Pratiques géometrales*, the Privilège of which is dated Mai 16, 1653. I believe Bosse was expelled from the French Academy of Painting and Sculpture in 1666, on account of a tract he published.

However, in Joh. Christophorii Sturmii *Mathesis Juvenilis*, Altdorff, 1702, p. 233, the method of De Ville is ascribed to Carolus Renaldinus, *De Resol. et Comp. Mathem.*, p. 367. Christian Wolff, in *Elementa Matheseos Universæ*, also makes this reference. Renaldinus was, I believe, born in 1612. If so, I fancy the problem must be due to De Ville, and its present form to Renaldinus. De Ville's book on *Fortifications* must have been of some value, as I find frequent allusions to it. The Marburg copy formerly belonged to one of the Hessen Princes. In Adams's *Mathematical Instruments* (1803), instead of making the perpendicular $\sqrt{3}$ times the radius, it is made 1·75 times the radius, which brings the pentagon correct to about 1′ 30″. This construction is probably taken from Hawnes's *Mensuration*. The date of this I cannot ascertain; but I believe his *Arithmetic* reached a ninth edition in 1745.

10675. (R. Knowles, B.A.)—Prove that the four points of intersection of the opposite sides of an octagon, inscribed in a conic, are collinear.

Solution by J. H. Grace.

Consider the case in which two pairs of opposite sides of the octagon are parallel; then, if this theorem be true, the other pairs of sides must always be parallel. This is a sufficient indication that the theorem is not, in general, true.

In the proof given on p. 114 of Vol. lvi., it is *assumed* that the diagonals of the hexagon are concurrent.

11519. (Professor Farjon.)—On détermine le plan radical des sphères ayant pour diamètres deux arêtes opposées d'un tétraèdre; les trois plans ainsi obtenus se coupent en un certain point. Démontrer que ce point et le centre de la sphère circonscrite au tétraèdre sont symétriques par rapport au centre de gravité du tétraèdre.

Solution by Professor Nash, M.A.

Taking one vertex as origin, let α, β, γ, δ, ϵ be the vectors to the other vertices, the centre of gravity, and the centre of the circum-sphere.

The equations of one pair of spheres, and that of their radical plane, are

$$\rho^2 - S\alpha\rho = 0, \quad \rho^2 - S\rho(\beta+\gamma) + S\beta\gamma = 0, \quad S\rho(\beta+\gamma-\alpha) = S\beta\gamma;$$

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hence the vector ρ of the intersection of the three planes is given by

$$\rho S(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma) = \Sigma S\beta\gamma\, V(\gamma+\alpha-\beta)(\alpha+\beta-\gamma),$$

or $$4\rho S\alpha\beta\gamma = 2\Sigma S\beta\gamma V\alpha(\beta-\gamma) - 2\Sigma V\beta\gamma S\alpha(\beta+\gamma).$$

Also $$2\epsilon S\alpha\beta\gamma = \Sigma a^2 V\beta\gamma;$$

$\therefore$ $$2(\rho+\epsilon) S\alpha\beta\gamma = \Sigma V\beta\gamma S\alpha(\alpha+\beta+\gamma) = (\alpha+\beta+\gamma) S\alpha\beta\gamma = 4\delta S\alpha\beta\gamma;$$

therefore $$\rho+\epsilon = 2\delta.$$

9983. (Professor Ignacio Beyens.)—Si (S) désigne la somme des surfaces des cercles de Neuberg correspondantes à chacun des côtés d'un triangle ABC, (e) l'excentricité de l'ellipse de Brocard, on aura

$$S = \pi(a^2+b^2+c^2)\, e^2/(1-e^2).$$

Solution by Professor G. B. M. Zerr.

The equations to the circles of Neuberg for a, b, c, are

$$x^2+y^2-ay\cot\omega+\tfrac{3}{4}a^2 = 0, \text{ \&c., \&c.;}$$

therefore the (radii)^2 are $\frac{1}{4}a^2(\cot^2\omega-3)$, $\frac{1}{4}b^2(\cot^2\omega-3)$, $\frac{1}{4}c^2(\cot^2\omega-3)$

therefore $S = \frac{1}{4}\pi(a^2+b^2+c^2)(\cot^2\omega-3)$; also $\sin^2\omega = \frac{1}{4}(1-e^2)$;

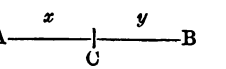
therefore $$S = \pi(a^2+b^2+c^2)\, e^2/(1-e^2).$$

11525. (Professor Shields.)—Two boats, A and B, a certain distance apart on a river, whose current rate is $2\frac{1}{2}$ miles an hour, agreed to run to each others' starting-points. Boat A, whose running rate in still water is 14 miles per hour, is 10 hours longer in running *up stream* from B to A than in running down the same distance from A to B; and boat B, whose running rate in still water is 8 miles an hour, is $41\frac{2}{7}$ minutes longer in running 8 miles *up* the river, than in running 8 miles *down* the same. They both start at the same time from their respective places (A 14 —— $2\frac{1}{2}$ —— B 8), and run to each others' starting-points (the fastest boat, A, running down stream, while B runs up). Find (1) the entire distance run by the boats, and (2) how far each boat was from its starting-point when they met or passed.

Solution by J. N. Bissell; J. C. Strachan; *and others.*

The stream running from A to B, let

$$x = AC, \quad y = BC;$$



then, the time from starting until the meeting of

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each being the same, we have

$$\frac{x}{14+2\frac{1}{2}} = \frac{y}{8-2\frac{1}{2}}, \text{ and } \frac{11x}{2} = \frac{33}{2}y, \text{ or } x = 3y;$$

whence $y = 94\frac{7}{8} = BC$, and $x = 3 \times 94\frac{7}{8} = 284\frac{5}{8} = AC$.

Hence whole distance $= 379\frac{1}{2}$ from A to B, all in miles.

7561. (W. Perryman.)—If $xy = 1$ be the equation to the rectangular hyperbola, prove that the difference between the length of the infinite arc of the curve from its vertex, and that of its asymptote from the centre, is equal to $\frac{2}{3}.\frac{4}{3}.\frac{6}{7}.\frac{8}{9}.\frac{10}{11}.\frac{12}{11}.\frac{14}{15}.\frac{16}{15}.\frac{18}{19}.\frac{20}{19}\ldots$, and its value to the fifteenth decimal place is ·847213084793973; or, if the curve be severed at vertex and be made to coincide with the asymptote by becoming straight, the point of severance will fall short of the centre of the curve by the above value.

Solution by Professor G. B. M. Zerr.

Turn the axis through an angle $\frac{1}{4}\pi$, and the equation to the rectangular hyperbola becomes $x^2 - y^2 = 2$; the perpendicular from the centre on a tangent is $p^2 = 2\cos 2w$; and, when the tangent is the asymptote, w becomes $\frac{1}{4}\pi$; hence the difference between the asymptote and its infinite arc measured from the vertex is given by the definite integral

$$\int_0^{\frac{1}{4}\pi} (2\cos 2w)^{\frac{1}{2}}\, dw.$$

Let $2w = \theta$, then $d\theta = 2dw$; therefore

$$2^{\frac{1}{2}} \int_0^{\frac{1}{4}\pi} (\cos 2w)^{\frac{1}{2}}\, dw = \frac{1}{2^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} (\cos\theta)^{\frac{1}{2}}\, d\theta = \tfrac{1}{2}\,(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, \Gamma(\tfrac{3}{4})/\Gamma(\tfrac{5}{4}).$$

Hence the difference between the two asymptotes and both infinite arcs, measured from the vertex, is $(\frac{1}{2}\pi)^{\frac{1}{2}}\, \Gamma(\frac{3}{4})/\Gamma(\frac{5}{4})$.

$$\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{5}{4}) = -\frac{5.9.13.17.21.25.29\ldots}{3.7.11.15.19.23.27\ldots},$$

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}} = \pm\frac{2.4.6.8.10.12.14.16.18.20\ldots}{3.5.7.9.11.13.15.17.19.21\ldots};$$

therefore

$$(\tfrac{1}{2}\pi)^{\frac{1}{2}}\, \Gamma(\tfrac{3}{4})/\Gamma(\tfrac{5}{4})$$

$$= \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{11}\cdot\frac{14}{15}\cdot\frac{16}{15}\cdot\frac{18}{19}\cdot\frac{20}{19}\cdot\frac{22}{23}\cdot\frac{24}{23}\cdot\frac{26}{27}\cdot\frac{28}{27}\cdots$$

$$= \cdot 847213084793979 \text{ to the fifteenth decimal.}$$

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11452. (Professor De Longchamps.)—Resoudre les equations
$(y-z+b^2-c^2)^2 = 4a^2x$, $(z-x+c^2-a^2)^2 = 4b^2y$, $(x-y+a^2-b^2)^2 = 4c^2z$.

Solution by Professor Cayley, F.R.S.

Regarding (x, y, z) as rectangular coordinates, the three equations represent parabolic cylinders having the plane infinity for a common tangent plane at the point $x = y = z$; any two of them meet in a quadriquadric curve, having the point in question for a node, and, since at this point the third surface has the same tangent plane, it touches there each of the two branches of the curve, and the point in question reckons therefore as four points of intersection; there remain therefore four points of intersection at a finite distance. It is at once seen that the equations are satisfied by the values $x, y, z = (b-c)^2, (c-a)^2, (a-b)^2$; and, since the equations contain only a^2, b^2, c^2, we thence at once deduce three other solutions, viz., we have the four following values of x, y, z:—

$$(b-c)^2,\ (c-a)^2,\ (a-b)^2;\quad (b-c)^2,\ (c+a)^2,\ (a+b)^2;$$
$$(b+c)^2,\ (c-a)^2,\ (a+b)^2;\quad (b+c)^2,\ (c+a)^2,\ (a-b)^2.$$

[Another Solution is given on p. 128 of Vol. LVII.]

11648. (Professor Sylvester.)—Show that the complete integrals of the equations in differences $\dfrac{u_{x+1}-3u_x}{u_x-3u_{x-1}} = \dfrac{u_x^3}{u_{x-1}^3}$, $\dfrac{v_{x+1}+3v_x}{v_x+3v_{x-1}} = \dfrac{v_x^3}{v_{x-1}^3}$, is

$$u_x = C(a^{3^x}-a^{-3^x}), \qquad v_x = C(a^{3^x}+a^{-3^x}).$$

Solution by H. W. Curjel, B.A.

$$\frac{u_{x+1}-3u_x}{u_x^3} = \frac{u_x-3u_{x-1}}{u_{x-1}^3} = \text{a constant} = \frac{1}{C^2} \text{ say};$$

hence
$$u_x^3 = (u_{x+1}-3u_x)\,C^2;$$

thus, if
$$u_x = C\left(w_x - \frac{1}{w_x}\right),$$

then
$$C^3\left(w_x-\frac{1}{w_x}\right)^3 = C^3\left(w_{x+1}-\frac{1}{w_{x+1}}-3w_x+\frac{3}{w_x}\right);$$

hence
$$w_x^3-\frac{1}{w_x^3} = w_{x+1}-\frac{1}{w_{x+1}};$$
$$w_x = a^{3^x} \quad \text{or} \quad (-1)^x a^{(-3)^x};$$

therefore in both cases
$$u_x = C(a^{3^x}-a^{-3^x}).$$

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Again, $\dfrac{v_{x+1}+3v_x}{v_x^3} = \dfrac{v_x+3v_{x-1}}{v_{x-1}^3} = \text{a constant} = \dfrac{1}{C^3}$ say;

then $v_x^3 = C^3(v_{x+1}+3v_x)$;

thus, if $v_x = C\left(w_x + \dfrac{1}{w_x}\right)$,

$$C^3\left(w_x+\frac{1}{w_x}\right)^3 = C^3\left(w_{x+1}+\frac{1}{w_{x+1}}+3w_x+\frac{3}{w_x}\right);$$

hence $w_x^3+\dfrac{1}{w_x^3} = w_{x+1}+\dfrac{1}{w_{x+1}}$;

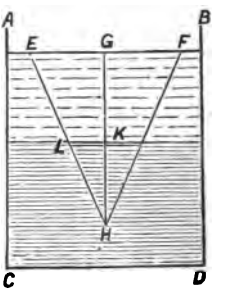
$w_x = a^{3^x}$ or $a^{(-3)^x}$;

therefore in both cases $v_x = C(a^{3^x}+a^{-3^x})$.

11630. (Professor DE WACHTER.)—Un cône circulaire droit, formé d'une matière homogène de densité d, est entièrement immergé dans un vase contenant deux liquides superposés de densités δ_1 et δ_2 ($\delta_1 > \delta_2$). Montrer que le plan de séparation des deux liquides divise l'axe du cône dans un rapport égal à $\left(\dfrac{\delta_1-\delta_2}{d-\delta_2}\right)^{\frac{1}{3}}-1$.

Solution by Professor ZERR; H. W. CURJEL, B.A.; *and others.*

Let ABDC = the vessel, EFH the cone, GH its altitude, EG = r, GH = a, KH = x, LK = y; then $y = \dfrac{rx}{a}$; therefore



$$\frac{\pi}{3}r^2\left(\frac{x^3}{a^2}\delta_1+a\delta_2-\frac{x^3}{a^2}\delta_2\right) = \frac{\pi}{3}r^2ad,$$

or $x = a\left(\dfrac{d-\delta_2}{\delta_1-\delta_2}\right)^{\frac{1}{3}}$,

$\dfrac{a}{x} = \left(\dfrac{\delta_1-\delta_2}{d-\delta_2}\right)^{\frac{1}{3}}$, $\dfrac{a}{x}-1 = \left(\dfrac{\delta_1-\delta_2}{d-\delta_2}\right)^{\frac{1}{3}}-1$;

therefore $\dfrac{a-x}{x} = \left(\dfrac{\delta_1-\delta_2}{d-\delta_2}\right)^{\frac{1}{3}}-1$.

11616. (Professor MINCHIN, M.A.)—Of five coplanar forces in equilibrium, given the lines of action of all, the magnitude of one, and the ratio of two others, find the magnitudes of all by a very simple geometrical construction.

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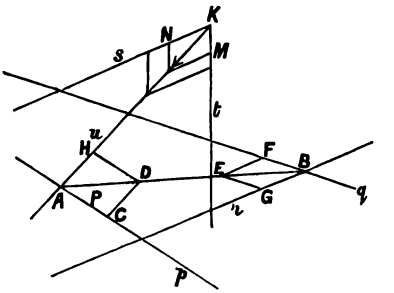
Solution by H. W. CURJEL, B.A.

Let P, Q, R, S, T, be the forces, P being given in magnitude, and the ratio of S and T being given. Since the ratio of S and T is given, the line of action of their resultant, U say, may be found by the parallelogram of forces. Let the lines of action of the forces be p, q, r, s, t, u. The resultants of P and U, and of R, Q are in equilibrium, and must both therefore pass through A and B, the points of intersection of p and u, and r and q.



Hence, if we cut off AC from p = P, and draw CD parallel to u cutting AB in D, and from BA cut off BE = DA, and draw EF parallel to r cutting q in F, and EG parallel to q cutting r in G, BF = Q, BG = R. Again, drawing DH parallel to p cutting u in H, and cutting KL off from KA, = HA, where K is point of intersection of s and t, and drawing LM, LN parallel to s and t, cutting t and s in M and N, we get KN = S and KM = T.

11666. (The EDITOR.)—Construct a triangle, having given the orthocentre P, the point Q where the circumcircle is cut by the external bisector of the vertical angle A, the angle AQP, and the difference of the angles B, C, at the base of the triangle.

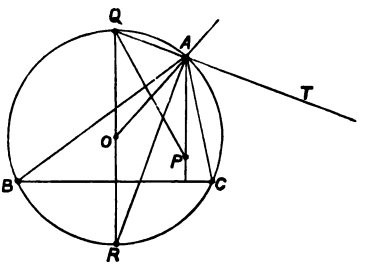
Solution by R. F. DAVIS, M.A.

Let R be the point where the circumcircle is cut by the internal bisector of the vertical angle A; so that QR passes through the circumcentre O, and bisects BC at right angles. Then we have

angle PAR = $\frac{1}{2}$OAP

$= \frac{1}{2}(B \sim C) = \theta$ say,

and is known.



At the point Q in the straight line PQ make the angle PQT equal to the given angle AQP; upon QT take a point A, such that the angle PAT is the complement of θ. Draw QR parallel to AP, and make OQ = OA. This fixes O; whence, &c.

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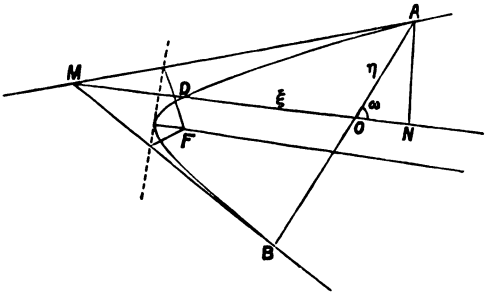
11625. (Professor DE LONGCHAMPS.) — Soient MA, MB deux tangentes à une parabole P. Le demi-paramètre p de cette courbe est donné par la formule $p \cdot MO^3 = (\text{aire } MAB)^2$, O désignant le milieu de AB. En utilisant cette relation, montrer que le lieu des foyers des paraboles de forme invariable, inscrites dans un angle fixe θ, est la quartique correspondant à l'équation $4x^2y^2\sin^4\theta = p^2(x^2+y^2+2xy\cos\theta)$.

Solution by J. BURKE, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

Let F be the focus, AN the perpendicular from A on MO, $\omega = \angle AON$, MD = DO = ξ, AO = η, π_1 and π_2 the perpendiculars from F on MA



and MB respectively, MA = a, MB = b; x, y the coordinates of F relative to MA and MB as axes.

Then, if p' be the semi-parameter at D,

$$p' = \frac{p}{\sin^2\omega} = \frac{\eta^2}{2\xi}; \qquad \therefore \quad p = \frac{(\eta\sin\omega)^2}{2\xi};$$

but
$$\eta\sin\omega = \frac{1}{2\xi}(ab\tfrac{1}{2}\sin\theta);$$

$$\therefore \quad p = \frac{(ab\tfrac{1}{2}\sin\theta)^2}{8\xi^3} = \frac{(\text{area AMB})^2}{MO^3}.$$

Again,
$$n\sin\theta = \pi_1 = \tfrac{1}{2}p \cdot \frac{a}{\eta\sin\omega} = \frac{2p\xi}{b\sin\theta}, \text{ by } (\alpha);$$

$$\therefore \quad \frac{b}{\xi} = \frac{2p}{n\sin^2\theta}; \qquad \text{similarly} \quad \frac{a}{\xi} = \frac{2p}{y\sin^2\theta}.$$

Hence
$$\frac{a^2+b^2+2ab\cos\theta}{\xi^2} = \frac{4p^2}{\sin^4\theta}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{2\cos\theta}{xy}\right);$$

but it can be easily shown that

$$16\xi^2 = a^2+b^2+2ab\cos\theta;$$

therefore
$$4x^2y^2\sin^4\theta = p^2(x^2+y^2+2xy\cos\theta).$$

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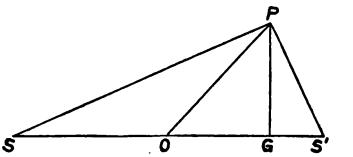
11633. (D. T. Griffiths.)—S and S′ are the foci of an oval of Cassini, and P is any point on the curve. If the normal at P meets SS′ at G, prove that $SG : S'G = SP^2 : S'P^2$.

Solution by H. W. Curjel, B.A.; S. Dale; *and others.*

Bisect SS′ in O, and make

$$\angle S'PG = \angle SPO;$$

then PG is the normal at P.



Also, $$\frac{SG}{SP} = \frac{\sin SPG}{\sin SGP}$$

and $$\frac{S'G}{S'P} = \frac{\sin S'PG}{\sin S'GP};$$

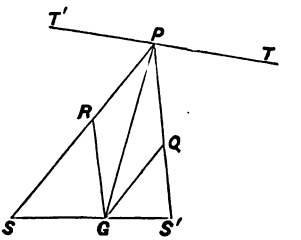
$$\therefore \quad \frac{SG}{S'G} = \frac{SP}{S'P}\,\frac{\sin SPG}{\sin S'PG};$$

Again, $$\frac{SP}{SO} = \frac{\sin SOP}{\sin SPO} \quad \text{and} \quad \frac{S'P}{S'O} = \frac{\sin S'OP}{\sin S'PO};$$

hence $$\frac{SP}{S'P} = \frac{\sin S'PO}{\sin SPO} = \frac{\sin SPG}{\sin S'PG};$$

$$SG : S'G = SP^2 : S'P^2.$$

[*Otherwise*:—If $PS = r_1$, $PS' = r_2$, the equation to the oval is $r_1 r_2 - \text{const.} = 0 = F(r_1 r_2)$; and if TT′ be the tangent at P, PG the normal, GQ parallel to PS, GR parallel to PS′, then we have



$$PR : PQ = \cos TPS' : -\cos TPS$$

$$= \frac{dF}{dr_1} : \frac{dF}{dr_2} = r_2 : r_1 = S'P : SP;$$

$$\therefore \quad PQ : PR = SP : S'P;$$

but $$PQ : SG = S'P : SS',$$

and $$PR : S'G :: SP : SS';$$

$$\therefore\ PQ : PR = SG \,.\, S'P : S'G \,.\, SP = SP : S'P; \quad \therefore\ SG : S'G = SP^2 : S'P^2.]$$

11681. (D. Biddle.)—A hollow cylinder, of length l, and intermediate radius r, revolves on its axis with an angular velocity v. A particle is proceeding within it with a uniform velocity V, parallel to the axis, but otherwise at random. In the plane of the axis, and extending to the circumference, is a partition. Find the probability that the particle and the partition will collide, (1) when the partition extends right across the cylinder, (2) when it is on one side of the axis only. Also give the mean distance at which the collision occurs in each case.

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Solution by H. W. CURJEL, B.A.

The time during which the particle is within the cylinder $= l/V$.
The time taken by one revolution $= 2\pi/v$.

Therefore (1) chance $= \frac{l}{V} : \frac{\pi}{v} = \frac{lv}{\pi V}$; (2) chance $= \frac{l}{V} : \frac{2\pi}{v} = \frac{lv}{2\pi V}$.

If (1) $l/V > \pi/v$, or (2) $l/V > 2\pi/v$, the collision is certain in the corresponding case.

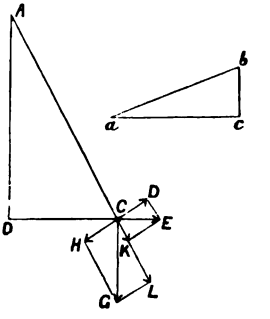
The mean position of the collision is seen by symmetry to be in the plane bisecting the axis of the cylinder at right angles, except in the cases specified in the preceding paragraph, when the mean positions are (1) $\frac{\pi V}{2lv}$ and (2) $\frac{\pi V}{lv}$ from the entrance of the cylinder.

Mean distance from the axis of the cylinder is

$$\int_0 2\pi x^2 dx \Big/ \int_0^r 2\pi x dx = \tfrac{2}{3}r.$$

5785. (D. EDWARDES.)—A train goes round a curve, the radius of the curve lying midway between the metals being r. If m be the mass of the train, v its velocity, and a the breadth of the gauge, find the pressure on the metals, the outer metal being raised so that the pressure is perpendicular to the floor of the carriage.

Solution by Professor G. B. M. ZERR.



In the figure, let C, the bob of the conical pendulum, represent the train. Then $CG = m$; also, let the centrifugal force $CE = f$. Resolve CG into its components CH perpendicular to, and CL parallel to AC; also, CE into its components $CD = CH$ perpendicular to, and CK parallel to AC. Then pressure $= P = CL + CK$;

$$\therefore \quad P = M\cos y + f\sin y,$$

where $y = \angle OAC$.

But $f = mv^2/gr$, $v^2 = gr^2/h$,

where $h = AO$, $r = OC$, $\sin y = r/(r^2+h^2)^{\frac{1}{2}}$, $\cos y = h/(r^2+h^2)^{\frac{1}{2}}$;

$$\therefore \quad P = (m/h)(r^2+h^2)^{\frac{1}{2}}.$$

Now let abc be a section of the road bed; then acb is similar to AOC, $ab = a$, $bc = e$, $ac = w$. Then $(r^2+h^2)^{\frac{1}{2}}/h = a/w$, but $w = a\cos y$. From the triangle CEG we get $\tan y = v^2/gr$, $\cos y = gr/(g^2r^2+v^4)^{\frac{1}{2}}$;

$$\therefore \quad \left[(r^2+h^2)/h^2\right]^{\frac{1}{2}} = \sec y = (g^2r^2+v^4)^{\frac{1}{2}}/gr; \quad \text{and} \quad P = m/gr \,.\, (g^2r^2+v^4)^{\frac{1}{2}}.$$

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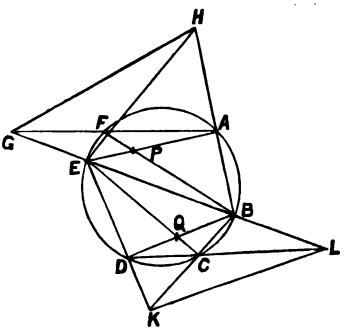
10746. (R. KNOWLES, B.A.)—A hexagon is inscribed in a conic; prove that the points of intersection of the diagonals of the two quadrilaterals formed by the sides of the hexagon and one of its diagonals, and the point of intersection of the diagonals of the corresponding circumscribed hexagon, are collinear.

Solution by J. H. GRACE.

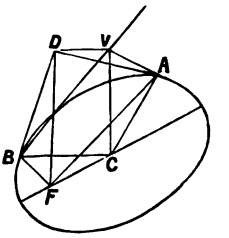


Consider the simplest possible case, viz., that in which a hexagon inscribed in a circle has its opposite sides parallel. Let ABCDEF be the hexagon and BE the diagonal in question. Let P and Q be the points of intersection of the diagonals. Then (see figure) P is the pole of GH, and Q of KL.

If the theorem be true, PQ must pass through the centre of the circle; hence GH and KL must be parallel. Thus, since BHEK is a parallelogram, it plainly follows that BC = EF, or the opposide sides of the hexagon must be equal. Hence, by projection, the necessary and sufficient condition for the truth of the theorem is that the diagonals must be concurrent. [See Vol. LVI., p. 88, for another attempt.]

11173. (Professor LAISANT.)—Soient CA, CB deux demi-diamètres conjugués d'une ellipse, et F l'un des foyers; on construit le parallélogramme FAVB. Démontrer que (1) la droite CV est dirigée suivant la bissectrice de l'angle AVB; (2) $CV^2 = 2VA \cdot VB$; (3) le lieu du point V est une ellipse quand on fait varier le système des diamètres conjugués. Étendre ces propriétés à l'hyperbole.

Solution by W. J. GREENSTREET, M.A.; M. BRIERLEY; *and others.*



1. Draw BD, DA respectively parallel to CA, CB; then the triangles DBF, AVC are equal in all respects, and $\angle AVC = BFD = \frac{1}{2}BFA = \frac{1}{2}AVB$; hence VC bisects the angle AVB.

3. And VC is equal and parallel to DF; hence VD is equal and parallel to CF; thus the loci of D, V are an ellipse similar to the given ellipse and an ellipse equal to the locus of D.

2. If A, B are ϕ, $-\phi$, then D is

$$a(\cos\phi - \sin\phi),\ b(\sin\phi + \cos\phi),$$

and $VC^2 = FD^2 = 2(a + c\sin\phi)(a - c\cos\phi)$

$$= 2VB \cdot VA.$$

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In the hyperbola D lies on an asymptote; and it is easily seen that, as AF is parallel to BV and AC parallel to DB, therefore DV is equal and parallel to FC; therefore locus of V is a line through the second focus F′ parallel to CM. Also FD bisects the angle between the radii vectores from F to the contact points of tangents from D to the hyperbola; therefore it bisects the angle between FA and a parallel through F′ to CD; therefore CV bisects F′VD.

Finally, if A and B are $(a\sec\phi,\ b\tan\phi)$, $(a\tan\phi,\ b\sec\phi)$,

D is $a(\sec\phi+\tan\phi),\ b(\sec\phi+\tan\phi)$;

$$\therefore\quad CV^2 = FD^2 = \left[a(\sec\phi+\tan\phi)-c\right]^2+b^2\left[\sec\phi+\tan\phi\right]^2$$
$$= 2c(\sec\phi+\tan\phi)(c\sec\phi-a) = 2AV\,.\,F'V.$$

11550. (Professor Ramaswami Aiyar, M.A.)—Prove that, if triangles inscribed in a circle X may be circumscribed to a given conic, the locus of their orthocentres is a circle Y; and triangles may be inscribed in Y which circumscribe the conic; the locus of *their* orthocentres is X.

Solution by J. Burke, M.A.; Professor Ramchundra; *and others.*

Taking the centre of X as origin, and any two rectangular lines through it as axes, the equation of circle is

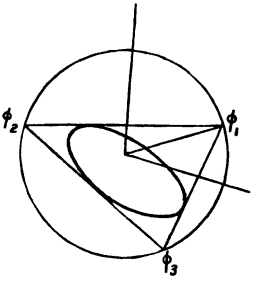
$$x^2+y^2=r^2;$$

and, if the tangential equation of the conic be

$$A\lambda^2+B\mu^2+b\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0,$$

and ϕ_1, ϕ_2, ϕ_3 the angles the radii vectores to vertices of Δ make with axis of x, the equation of join $(\phi_2\phi_3)$ is

$$x\cos\tfrac{1}{2}(\phi_2+\phi_3)+y\sin\tfrac{1}{2}(\phi_2+\phi_3) = r\cos\tfrac{1}{2}(\phi_2-\phi_3).$$



Hence, substituting

$$\lambda=\cos\tfrac{1}{2}(\phi_2+\phi_3),\quad \mu=\sin\tfrac{1}{2}(\phi_2+\phi_3),\quad \nu=-r\cos\tfrac{1}{2}(\phi_2-\phi_3)$$

in tangential equations of conic, we get

$$A\left[1+\cos(\phi_2+\phi_3)\right]+B\left[1-\cos(\phi_2+\phi_3)\right]+br^2\left[1+\cos(\phi_2-\phi_3)\right]$$
$$-2Fr\left[\sin\phi_2+\sin\phi_3\right]-2Gr\left[\cos\phi_2+\cos\phi_3\right]+2H\sin\left[\phi_2+\phi_3\right]=0.$$

Put $l=\cos\phi_1+\cos\phi_2+\cos\phi_3$, $m=\sin\phi_1+\sin\phi_2+\sin\phi_3$; then

$$\theta=\phi_1+\phi_2+\phi_3,\quad \cos(\phi_2-\phi_3)=\tfrac{1}{2}\left[l^2+m^2-1\right]-l\cos\phi_1-m\sin\phi_1,$$

$$\cos\phi_1\left[(A-B)\cos\theta+2H\sin\theta-Clr^2+2Gr\right]$$
$$+\sin\phi_1\left[(A-B)\sin\theta-2H\cos\theta-Cmr^2+2Fr\right]$$
$$+\tfrac{1}{2}br^2\left[l^2+m^2-1\right]-2Fmr-2Glr+A+B+br^2=0,$$

or

$$P\cos\phi_1+Q\sin\phi_1+R=0,$$

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which holds also for ϕ_2 and ϕ_3, when $P = 0$, $Q = 0$, $R = 0$. These are two independent equations, and $P = 0$, $Q = 0$ must be fulfilled by ϕ_1, ϕ_2, ϕ_3; while from R we easily get the required locus.

Let (x), (y) be coordinates of CG of Δ, and x', y' those of the orthocentre; then $(x) = \frac{1}{3}x'$, $(y) = \frac{1}{3}y'$; but $(x) = \frac{1}{3}rl$, $(y) = \frac{1}{3}rm$; therefore $x' = rl$, $y' = rm$. Substituting these values of l and m in $R = 0$, the locus is $x^2 + y^2 - 2\left[2F/b\right]y - 2\left[2G/b\right]x + 2\left[(A + B)/b + \frac{1}{2}r^2\right] = 0$,

The coordinates of the centre of the conic are G/b and F/b. Hence centre of conic bisects join of centres of X and Y.

Let R = radius of Y, then

$$R^2 = 4\frac{G^2}{b^2} + 4\frac{F^2}{b^2} - \frac{2(AC + BC)}{b^2} - r^2.$$

But we know that equation of director circle of conic is

$$b\left[x^2 + y^2\right] - 2Gx - 2Fy + A + B = 0;$$

therefore, if a and b be lengths of axes, we have

$$a^2 + b^2 = (G^2 + F^2 - AC - BC)/b^2;$$

$$\therefore \quad R^2 = 2\left[a^2 + b^2\right] + 2\delta^2 - r^2; \qquad \therefore \quad R^2 + r^2 = 2\left[a^2 + b^2 + \delta^2\right];$$

where δ = distance of centre of conic from origin. The conclusion follows.

11108. (R. Tucker, M.A.)—(1) OPQ is a triangle formed by the intersections of a transversal PQ with the asymptotes OP, OQ of a hyperbola. If $\left.\begin{matrix} OP = h, \ OQ = k, \\ \angle POQ = 2\alpha, \end{matrix}\right\}$ prove that, if the "S." point of the triangle be on the curve, then $h^3k^3 = \lambda^2[h^2 + k^2 - hk\cos 2\alpha]^2$, where $\lambda^2 = a^2 + b^2$. Find also the locus of the centroid and circumcentre. (2) OPQ is a triangle formed by the tangents OP, OQ to a parabola and the chord of contact PQ: prove that, if the "S." point of the triangle is on the curve, then $PQ^2 = 2\,OP \cdot OQ$.

Solution by the Proposer.

1. The equation, referred to the asymptotes, is

$$4xy = a^2 + b^2 = \lambda^2;$$

also, $\qquad PQ = l, \quad l^2 = h^2 + k^2 - 2hk\cos 2\alpha;$

and the equation to the "S." line through O is $hx - ky = 0$ ..... (1).

Let the "S." lines through P, Q cut off from OQ, OP the intercepts y_2, x_1; then $\qquad x_1/h = k^2/(k^2 + l^2), \quad y_2/k = h^2/(h^2 + l^2);$

$$\therefore \quad x_1 = hk^2/(h^2 + 2k^2 - 2hk\cos 2\alpha) \text{ ..... (2)};$$

$$y_2 = h^2k/(2h^2 + k^2 - 2hk\cos 2\alpha), \quad x/x_1 + y/k = 1, \quad x/h + y/y_2 = 1.$$

Hence, with (2), we get, rejecting the factor $h^2 + k^2 - 2hk\cos 2\alpha$,

$$2x\left[h^2 + k^2 - hk\cos 2\alpha\right] = hk^2, \quad 2y\left[h^2 + k^2 - hk\cos 2\alpha\right] = h^2k \text{ ... (3)}.$$

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From (1), $x/k = y/h = \lambda/2(hk)^{\frac{1}{2}}$;

hence, substituting in (3), $h^3k^3 = \lambda^2\left[h^2+k^2-hk\cos 2\alpha\right]^2$ ..... (4).

For a given value of h we get, by (4), four values of k, and so four "S." points also lie on a quartic curve. The loci of the centroid and circumcentre are, by (4), sextic curves.

2. Referring the parabola to its tangents, we have the equation

$$(x/h)^{\frac{1}{2}} + (y/k)^{\frac{1}{2}} = 1 \quad \text{..... } (a).$$

Equations (1), (2), (3) above hold in this case also, and

$$x/k = y/h = hk/(h+k)^2, \quad \text{from } (a).$$

Hence, substituting in (3), we get

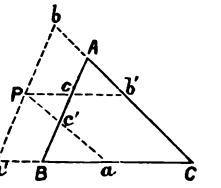
$$(h+k)^2 = 2\left[h^2+k^2-hk\cos 2\alpha\right],$$

i.e., $h^2+k^2-2hk\cos 2\alpha = 2hk$, or $PQ^2 = 2\,OP\,.\,OQ$.

11598. (W. J. Greenstreet, M.A.)—Through any point on the axis of homology of a triangle ABC and its pedal, parallels are drawn to AB, BC, CA. The six points where these parallels meet the sides lie on a conic.

Solution by J. E. St. Clair; Professor Chakravarti; *and others.*

Let three lines, drawn through any point P parallel to the sides of the triangle ABC, meet BA, AC, CB in cc', bb', aa', respectively. Then the opposite sides of the hexagon $aa'bb'cc'$, being parallel to each other, intersect on a line, namely, the line at infinity. Therefore the vertices lie on a conic.



[*Otherwise*:—If we multiply the three equalities, $Ab/Bb' = Cb/Ca'$, &c., we obtain Carnot's theorem (see Salmon's *Conics*, Art. 313).]

10675. (R. Knowles, B.A.)—Prove that the four points of intersection of the opposite sides of an octagon, inscribed in a conic, are collinear.

Note by H. W. Curjel, B.A.

The Proposer's solution, given on p. 114 of Vol. lvi., is not applicable to all cases, since, if AB, AH cut the line of collinearity of the intersections of opposite sides of the octagon in P and Q, then A, being dis-

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placed to A′ on the conic A′B, should pass through P, and A′H through Q according to the proposition; this is obviously impossible. The necessary and sufficient condition for the intersections to be collinear is that it should be possible to draw a conic touching the diagonals of the octagon and having double contact with the conic at the points where it meets the straight line determined by two of the intersections of opposite sides of the octagon.

11406. (Professor ZERR.)—Three points are taken at random in a sphere, and joined to the extremity of a fixed diameter; find the average volume of the tetrahedron thus formed.

Solution by the PROPOSER.

With the same notation as in Quest. 11126, Vol. LV., App. II., since

$$\text{volume} = \tfrac{1}{6}vw \sin(\phi+\psi)\, r(\cos\theta + \sin\lambda),$$

where PO is parallel to the plane which is perpendicular to the fixed diameter at its extremity, we have

$$\Delta = \frac{81}{64\pi^3 r^9}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\pi}\int_0^{2\pi}\int_0^{v'}\int_0^{w'} \tfrac{1}{6}vw\sin(\phi+\psi)\, r(\cos\theta+\sin\lambda) \times r\sin\theta\, d\theta\, 2\pi u\, du \sin(\theta+\psi)\, d\phi\, d\psi \sin\lambda\, d\lambda\, d\rho\, v^2\, dv\, w^2\, dw$$

$$= \frac{27}{16\pi^2 r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{v'} \sin\theta(\cos\theta+\sin\lambda)\, u^5\cos^4\psi\sin^2(\phi+\psi) \times \sin\lambda\, d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho\, v^3\, dv$$

$$= \frac{27}{4\pi^2 r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} \sin\theta(\cos\theta+\sin\lambda)\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi \times \sin\lambda\, d\theta\, du\, d\phi\, d\psi\, d\lambda\, d\rho$$

$$= \frac{27}{2\pi r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi} \sin\theta(\cos\theta+\sin\lambda)\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi \times \sin\lambda\, d\theta\, du\, d\phi\, d\psi\, d\lambda$$

$$= \frac{27}{4\pi r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} \sin\theta(4\cos\theta+\pi)\, u^9\sin^2(\phi+\psi)\cos^4\phi\cos^4\psi \times d\theta\, du\, d\rho\, d\psi$$

$$= \frac{9}{128\pi r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin\theta(4\cos\theta+\pi)\, u^9 \times (15\pi\cos^4\phi - 12\pi\cos^6\phi + 30\phi\cos^4\phi - 24\phi\cos^6\phi - 4\sin\phi\cos^7\phi + 30\sin\phi\cos^5\phi)\, d\theta\, du\, d\phi$$

$$= \frac{135\pi}{512 r^7}\int_0^{\frac{1}{2}\pi}\int_0^{u'} \sin\theta(4\cos\theta+\pi)\, u^9\, d\theta\, du$$

$$= \frac{27\pi r^3}{1024}\int_0^{\frac{1}{2}\pi}\sin''\theta(4\cos\theta+\pi)\, d\theta = \frac{9}{1024}\pi r^3 + \frac{3}{308}\pi^2 r^3.$$

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11491. (Professor SHIELDS.)—A. owned a tract of land in the form of an equilateral triangle. He had three sons, B., C., D.; three daughters, E., F., G.; and three grandchildren, H., I., J. Of this land, he gave to each son one of the three equal and largest possible circular tracts; to each daughter, one of the corner sections cut off by one of these circular tracts; and to each grandchild, one of the side sections cut off by two of these circular tracts; and for himself he reserved the tricuspid section between the three circular tracts. This section contained $(2500\sqrt{3}-3927)$ sq. rds., or 403·125+ sq. rds. Find the share of each son, daughter, and grandchild.

Solution by J. N. BISSELL; Prof. RADIKANSHUAN; *and others.*

Let GD $= r$;

then GH $= 2r$, AG $= r\sqrt{3}$;

hence $r^2(\sqrt{3}-\frac{1}{2}\pi) = 403{\cdot}125$,

$r^2 = 2500$ or $r = 50$ rods;

$\pi r^2 = 7854$ sq. rods = share of each son;

$r^2(\sqrt{3}-\frac{1}{3}\pi) = 1712{\cdot}125+$ sq. rds.

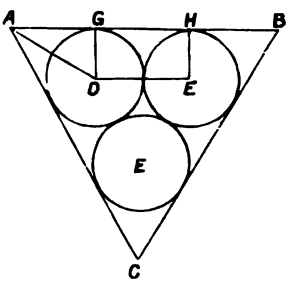
share of each daughter;

$r^2(2-\frac{1}{2}\pi) = 1073$ sq. rds.

share of each grandchild.

Area whole tract $= r^2(3+\sqrt{3})(1+\sqrt{3})$

$= 32220{\cdot}5$ sq. rds.



11375. (Professor ZERR.)—Find (1) the maximum rectangle, (2) the maximum parallelopiped, inscribed in, first, the curve $(x/a)^{\frac{1}{3}}+(y/b)^{\frac{1}{3}}=1$, second, the surface $(x/a)^{\frac{1}{3}}+(y/b)^{\frac{1}{3}}+(z/c)^{\frac{1}{3}}=1$, $a>b>c$.

Solution by H. W. CURJEL, B.A.

1. $4u \equiv 4xy$ is to be a maximum, with condition

$$(x/a)^{\frac{1}{3}}+(y/b)^{\frac{1}{3}}=1; \quad \text{therefore } du \equiv y\,dx+x\,dy=0,$$

and $$\frac{x^{-\frac{1}{3}}\,dx}{a^{\frac{1}{3}}}+\frac{y^{-\frac{1}{3}}\,dy}{b^{\frac{1}{3}}}=0; \quad \text{therefore } (x/a)^{\frac{1}{3}}=(y/b)^{\frac{1}{3}}=\tfrac{1}{2};$$

therefore $$4u = 4xy = 4ab\,(\tfrac{1}{2})^3 = 2ab.$$

2. $8u \equiv 8xyz$ is to be a maximum, with condition

$$(x/a)^{\frac{1}{3}}+(y/b)^{\frac{1}{3}}+(z/c)^{\frac{1}{3}}=1,$$

$$\frac{du}{xyz} \equiv \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0, \text{ and } \frac{x^{-\frac{1}{3}}\,dx}{a^{\frac{1}{3}}}+\frac{y^{-\frac{1}{3}}\,dy}{b^{\frac{1}{3}}}+\frac{z^{-\frac{1}{3}}\,dz}{c^{\frac{1}{3}}}=0;$$

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therefore $$dy\left(\frac{x}{y}-\frac{y^{-\frac{1}{3}}a^{\frac{1}{3}}}{x^{-\frac{1}{3}}b^{\frac{1}{3}}}\right)+dz\left(\frac{x}{z}-\frac{z^{-\frac{1}{3}}a^{\frac{1}{3}}}{x^{-\frac{1}{3}}c^{\frac{1}{3}}}\right)=0$$

therefore $$(x/a)^{\frac{1}{3}}=(y/b)^{\frac{1}{3}}=(z/c)^{\frac{1}{3}}=\tfrac{1}{3},$$

$$8u=8xyz=8abc\,(\tfrac{1}{3})^3=\frac{8\sqrt{3}\,abc}{243}.$$

11185. (R. W. D. Christie.)—Prove that

$$a^r+(a+d)^r\ \ldots\ldots\ \left\{a+(p-1)d\right\}^r=p\,(\mathrm{M}),$$

where $r=2$ or 5, and $p=$ any prime >3.

[*Ex. gr.*, $3^2+7^2+11^2+15^2+19^2+23^2+27^2=7\,(\mathrm{M})$,

also $2^5+4^5+6^5+8^5+10^5+12^5+14^5=7\,(\mathrm{M}).$]

Solution by H. J. Woodall.

Expanding, in powers of a, we get

$$a^r+(a+d)^r+(a+2d)^r+\ \ldots\ldots\ +\left\{a+(p-1)d\right\}^r$$

$$=p\,.\,a^r+r\,.\,da^{r-1}\Sigma q+\frac{r\,(r-1)}{1\,.\,2}d^2a^{r-2}.\,\Sigma q^2+\ \ldots\ +d^r\Sigma q^r;$$

the summations being all taken between the limits $q=1$ and $q=p-1$.

To prove that, under given conditions, $\Sigma q^n=p\,[\mathrm{M}]$, we have

$$p^{n+1}=\left\{1+(p-1)\right\}^{n+1}=1^{n+1}+(n+1)\,1^n(p-1)+\frac{(n+1)n}{1\,.\,2}1^{n-1}(p-1)^2+\ \ldots$$

$$+(n+1)\,1^1(p-1)^n+(p-1)^{n+1},\ \&c.,$$

$$(p-t)^{n+1}=\left\{1+(p-t-1)\right\}^{n+1},\ \&c.,\ \text{there being } p \text{ rows.}$$

Therefore, summing, we get

$$\Sigma(q+1)^{n+1}=p+(n+1)\,\Sigma q+\frac{(n+1)n}{1\,.\,2}\,\Sigma q^2+\ \ldots\ +(n-1)\,\Sigma q^n+\Sigma q^{n+1},$$

or $$p^{n+1}=p+(n+1)\,\Sigma q+\frac{(n+1)n}{1\,.\,2}\,\Sigma q^2+\ \ldots\ +(n-1)\,\Sigma q^n.$$

Hence, p being a prime, and if Σq^r is divisible by p from $r=1$ to $r=(n-1)$ inclusive, then $\Sigma q^n=p\,[\mathrm{M}]$, and the proposition is proved.

9545. (Professor Kalipada Banû, M.A.)—Prove that, when n is indefinitely increased, the area of the loop of the curve

$$a^{2n-2}y^2=n^3x^{2n}\frac{a-x}{a+x}\quad\text{is}\quad(\tfrac{1}{2}\pi)^{\frac{1}{n}}a^2.$$

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Solution by Professor G. B. M. ZERR.

$$y = n^{\frac{1}{2}}x^n\left[(a-x)/(a+x)\right]^{\frac{1}{2}}/a^{n-1}.$$

Let $x = a\sin\phi$; then we have

$$\tfrac{1}{2}A = a^2n^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\sin^n\phi(1-\sin\phi)\,d\phi = (\tfrac{1}{2}\pi)^{\frac{1}{2}}a^2, \text{ as } n \text{ approaches infinity.}$$

Therefore $$A = (2\pi)^{\frac{1}{2}}a^2.$$

5346. (W. E. WRIGHT, B.A.)—If $\dfrac{\sin\alpha}{\sin(\alpha-2\beta)} = c$, find $\int\sin\alpha\,d\beta$.

Solution by Professor G. B. M. ZERR.

Since $\sin\alpha = c\sin 2\beta/(1+c^2-2c\cos 2\beta)^{\frac{1}{2}}$, we have

$$\int\sin\alpha\,d\beta = c\int\frac{\sin 2\beta}{(1+c^2-2c\cos 2\beta)^{\frac{1}{2}}}\,d\beta = \tfrac{1}{2}(1+c^2-2c\cos 2\beta)^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\frac{\sin 2\beta}{\sin(\alpha-2\beta)} = \tfrac{1}{2}\frac{c\sin 2\beta}{\sin\alpha} = \pm\tfrac{1}{2}\left\{\sqrt{c^2-\sin^2\alpha}\mp\cos\alpha\right\}.$$

5646. (R. TUCKER, M.A.)—Normals to a lemniscate are drawn at points whose vectorial angles are θ and $\frac{1}{2}\pi+\theta$; find the locus of their intersections and the area of the curve.

Solution by Professor G. B. M. ZERR.

Let $r^2 = a^2\cos 2\theta$ be the equation to the lemniscate.
Then the equations to the normals are

$$x\sin 3\theta + y\cos 3\theta = a\sin 2\theta(\cos 2\theta)^{\frac{1}{2}},$$

$$x(\sin 3\theta+\cos 3\theta) + y(\cos 3\theta-\sin 3\theta) = -a\cos 2\theta(2\sin 2\theta)^{\frac{1}{2}}.$$

The coordinates of intersection are

$$x = a(\sin 2\theta\sin 3\theta - \sin 2\theta\cos 3\theta)(\cos 2\theta)^{\frac{1}{2}} - a\cos 2\theta\cos 3\theta(2\sin 2\theta)^{\frac{1}{2}},$$

$$y = a(\sin 2\theta\sin 3\theta + \sin 2\theta\cos 3\theta)(\cos 2\theta)^{\frac{1}{2}} + a\cos 2\theta\sin 3\theta(2\sin 2\theta)^{\frac{1}{2}};$$

hence the locus is $r^2 = a^2\sin 4\theta(\sin 2\theta + \cos 2\theta + \sqrt{\sin 4\theta})$,

and its area $$= 2a^2\int_0^{\frac{1}{2}\pi} r^2\,d\theta = \frac{4}{3}a^2.$$

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5323. (Rev. E. HILL, M.A.)—The radius of a thermometer bulb is ·3 of an inch, that of the stem ·01, height of the freezing point above the bulb 1 inch, and of the boiling point $7\frac{1}{2}$ inches; show that the expansion of mercury for a degree centigrade is about $\frac{1}{5550}$ of its volume.

Solution by H. J. WOODALL, A.R.C.S.

In this example the bulb is spherical.

Volume at 0° C. $= \frac{4}{3}\pi \times \cdot027 + 1 \times \pi \times \cdot0001$,

100° C. $= \frac{4}{3}\pi \times \cdot027 + 7\frac{1}{2} \times \pi \times \cdot0001$.

Let x = coefficient of expansion between 0° and 100° C.

$$100x = \frac{6\frac{1}{2} \times \cdot0001}{\frac{4}{3} \times \cdot027 + 1 \times \cdot0001} = \frac{\cdot00065}{\cdot03610} = \frac{65}{3610},$$

$$x = \frac{65}{361000} = \frac{1}{5550} \text{ about.}$$

5411. (A. MARTIN, LL.D.)—Find the probability that a random shot will hit a target a feet square at a distance of b feet.

Solution by H. W. CURJEL, B.A.

If we consider the shot as acted on by no forces and projected from a point in the perpendicular to the target through its centre, the chance that the elevation is right $= \theta/2\pi$, when $\tan \frac{1}{2}\theta = a/2b$.

Therefore the chance that the shot will hit the target

$$= \left(\frac{\theta}{2\pi}\right)^2 = \left(\frac{\tan^{-1} a/2b}{\pi}\right)^2.$$

10583. (J. MACNEILL, M.A.)—Find the contents, in gallons, of a circular-section basin, 6 feet in perpendicular height, the diameters being:—at base, 1 foot; at 1 foot perpendicular height, 2 feet; at 2 feet perpendicular height, 4 feet; and so on, the diameter being doubled for every foot of perpendicular height.

Solution by Professor G. B. M. ZERR.

The basin is generated by revolving about the axis of x the curve $y = 2^x$;

$$\therefore \quad v = \pi \int y^2 dx = \pi \int_0^6 2^{2x} dx = \frac{4095\pi}{2 \log 2} \text{ feet cubic} = 133167{\cdot}18403 \text{ gallons.}$$

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11484. (Professor ORCHARD, M.A., B.Sc.)—What relation must hold among the coefficients in the general cubic equation $x^3+px^2+qx+r=0$, in order that it may be represented as the product of two factors, one simple and one quadratic?

Solution by R. KNOWLES, B.A.; Professor IGNACIO BEYENS; *and others.*

$$x(x^3+px^2+qx+r) = \left[(x^2+\tfrac{1}{2}px+r/p)+\left\{(\tfrac{1}{4}p^2-q+2r/p)x^2+r^2/p^2\right\}^{\frac{1}{2}}\right] \times \left[(x^2+\tfrac{1}{2}px+r/p)-\left\{(\tfrac{1}{4}p^2-q+2r/p)x^2+r^2/p^2\right\}^{\frac{1}{2}}\right];$$

if $\frac{1}{4}p^2-q+2r/p=0$, this becomes

$$x(x+\tfrac{1}{2}p)(x^2+\tfrac{1}{2}px+2r/p);$$

and omitting x, the cubic is represented as the product of a simple and quadratic factor. This result shows that the cubic can be represented in this manner when one root equals the sum of the other two, whether the roots are all real or two of them imaginary.

11611. (MORGAN BRIERLEY.)—If X_1, X_2, X_3 denote the three perpendiculars from the three angles of a triangle ABC upon the opposite sides, and Y_1, Y_2, Y_3 perpendiculars upon the same from a given point within the triangle, prove that $\dfrac{Y_1}{X_1}+\dfrac{Y_2}{X_2}+\dfrac{Y_3}{X_3}=1$.

Solution by J. RICE; Professor ZERR; *and others.*

$$\frac{2\Delta\,.\,Y_1}{X_1}+\frac{2\Delta\,.\,Y_2}{X_2}+\frac{2\Delta\,.\,Y_3}{X_3}=a\,.\,Y_1+b\,.\,Y_2+c\,.\,Y_3=2\Delta;\quad\therefore\quad \Sigma\frac{Y_1}{X_1}=1.$$

11395. (J. L. MACKENZIE, B.A., B.Sc.)—Prove that $(x+y)^n-x^n-y^n$ is divisible by x^2+xy+y^2, when n is any odd number not a multiple of 3.

Solution by Profs. SITARAMAYAR, RADIKANSHUAN, *and others.*

It is the same if $(k+1)^n-k^n-1$ is divisible by k^2+k+1; and the proposition is proved if $(k+1)^n-k^n-1=0$ when $k^2+k+1=0$, which we proceed to show. Solving $k^2+k+1=0$, $k=\frac{1}{2}[-1\pm(-3)^{\frac{1}{2}}]$, where α and β are the imaginary roots of $k^3=1$, so that $\alpha^{3m}=\beta^{3m}=1$. It is obvious that whether $k=\alpha$ or β, $k+1=-\beta$ or $-\alpha$; and, as n is odd, $(k+1)^n-k^n-1$ becomes $-\alpha^n-\beta^n-1$. And, as n is of the form $6m\pm1$, the expression $=-(\alpha^{6m\pm1}+\beta^{6m\pm1}+1)$; $=-(\alpha^{\pm1}+\beta^{\pm1}+1)=0$, for $\alpha+\beta=-1$, and $\alpha\beta=1$.

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11667. (Ven. Archdeacon WILSON, M.A.)—When $4n+1$ is a prime number, it is an old and well-known property of numbers that it is expressible in the form of the sum of two squares. But the proofs throw little or no light on "the reason why." Can any connexion be shown, or any explanation given of this curious property? [Reference may be made to FERMAT, who announces the theorem in his *Observations on Diophantus*, VII.; to EULER, by whom it is proved in the *Nov. Com. Petr.*, V., p. 3; to SERRET, by whom it is proved in Art. 294 of his *Alg. Supérieure* (3rd edit., 1866); or to ROUSE BALL'S *Math. Recreations*.]

Note by R. F. DAVIS, M.A.

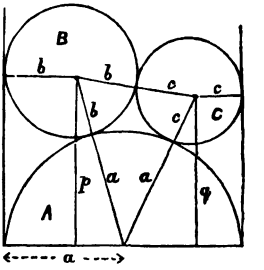
I am very much interested in Archdeacon WILSON'S question about the dissolution of a prime number into the sum of two squares, in regard to which I send the following note:—

Let $N = (4m+1)$; if N be not prime, there exists one solution at least of the Question, $N + x^2 = \square$.

If N be prime, according to the property of the question, there is always a solution of $N - x^2 = \square$. Hence, if we could prove, whether N be prime or not, there is always a solution of the equation, $N \sim p^2 = \square$, distinct from $p = 2m+1$, it would be sufficient. For, if $p^2 - N = q^2$, $N = p^2 - q^2$, and is therefore not a prime. On the contrary, if $N - p^2 = q^2$, then we shall have $N = p^2 + q^2$.

11703. (EDITOR.)—If three circles A, B, C touch each other two and two, and a common tangent to A and B is parallel to a common tangent of A and C, and if a, b, c be the radii, and p, q the distances of the centres of B and C from that diameter of A which is normal to the two parallel tangents, prove that $pq = 2a^2 = 8bc$.

Solution by the Rev. Dr. BRUCE; MORGAN BRIERLEY; *and others.*



Since $(a+b)^2 = (a-b)^2 + p^2$,

we have $p^2 = 4ab$.

Similarly, $q^2 = 4ac$.

Also $(b+c)^2 = (p \sim q)^2 + \{(a-b) + (a-c)\}^2$;

whence $pq = 2a^2$.

Thus $p^2q^2 = 4ab \cdot 4ac = 16a^2bc = pq \cdot 8bc$;

or $pq = 8bc$.

11704. (D. BIDDLE.)—Over the wheel of an Attwood's machine for illustrating the laws of gravitation passes a rope, to one end of which is

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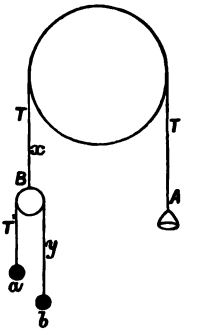
attached a weight A, and to the other a pulley B, over which play weights a and b. Find the positions of B and b at the end of each successive second, disregarding friction and the resistance of the air.

[This, in a general form, is the question of which Quest. 11389 was a particular instance.]

Solution by W. S. B. Woolhouse; H. W. Curjel, M.A.; *and others.*



Let A, a, b denote the values of the respective weights, and B the separate weight of the pulley; also T, T′ the tensions, and x, y the lengths of string, as represented in the diagram. Considering first the movement of the weight A, the velocity, estimated upwards, is $\frac{dx}{dt}$, the accelerative force is $\frac{d^2x}{dt^2}$, and the moving forces are the tension T and the force of gravity (W) in the opposite direction.

The equation of motion is therefore

$$\frac{A}{g} \cdot \frac{d^2x}{dt^2} = T - A \quad \text{.....(1).}$$

For the movement of the pulley B, separately considered, the velocity, estimated upwards, is also $\frac{dx}{dt}$, and the moving forces are $B + 2T' - T$.

We have therefore

$$\frac{W}{g} \cdot \frac{d^2x}{dt^2} = B + 2T' - T \quad \text{.....(2).}$$

Similarly the equations of motion of the weights a and b are respectively

$$\frac{a}{g}\left(\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\right) = T' - a \quad \text{.....(3),}$$

11704

and

$$\frac{b}{g}\left(\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2}\right) = b - T' \quad \text{.....(4).}$$

To eliminate the tensions T, T′, first add together the equations (1), (2) and twice (3), and afterwards add (3) and (4). Thus we get

$$\frac{A + B + 2b}{g} \cdot \frac{d^2x}{dt^2} + \frac{2b}{g} \cdot \frac{d^2y}{dt^2} = -A + B + 2b, \quad \frac{b-a}{g} \cdot \frac{d^2x}{dt^2} + \frac{b+a}{g} \cdot \frac{d^2y}{dt^2} = b - a.$$

From these we obtain the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and, by substitution in the preceding equations, also the values of the tensions T, T′ as hereunder stated,

$$\frac{d^2x}{dt^2} = g\,\frac{(b+a)(B-A) + 4ab}{(b+a)(B+A) + 4ab}, \quad \frac{d^2y}{dt^2} = \frac{2(b-a)A}{(b+a)(B+A) + 4ab};$$

$$T = 2A\,\frac{(b+a)B + 4ab}{(b+a)(B+A) + 4ab}, \quad T' = A\,\frac{4ab}{(b+a)(B+A) + 4ab}.$$

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By making $\beta = \frac{4ab}{a+b}$, and $B' = B + \beta$, these values are much simplified, and reduce down to the following:—

$$\frac{d^2x}{dt^2} = g\frac{B'-A}{B'+A}, \quad \frac{d^2y}{dx^2} = \frac{b-a}{b+a}\cdot\frac{2A}{B'+A},$$

$$T = \frac{2B'A}{B'+A}, \quad T' = \frac{\beta A}{B'+A}.$$

From these results we draw the following remarkable conclusions:—

1. Assuming $a = 0$, $b = 0$; then for the simple case of only two weights

$$\frac{d^2x^2}{dt^2} = g\frac{B-A}{B+A}, \text{ and } T = \frac{2AB}{A+B}.$$

2. Therefore, assuming another similar case in which the string connecting the two weights a and b is supposed to pass over a *fixed* pulley B, we should then have $T' = \frac{2ab}{a+b}$; and the whole weight on such fixed pulley would be $2T' = \frac{4ab}{a+b}$, which is identical with the value of β.

3. The movement of A and B would not be altered if the pulley B and suspended weights a, b were wholly removed and a simple weight B' $(= B + \beta)$ substituted for them.

This ingenious question presents a most interesting example of uniformly accelerated motion.

[The Proposer wishes to withdraw his note to the solution of Quest. 11389 (p. 50 of this Volume), as he is now convinced that Mr. Curjel's statement in the particular quoted is perfectly correct. In the last line of Mr. Curjel's solution, *read* $(2Ag - a - A)$ *for* $(2Ag - a - 1)$.]

11649. (Professor Lampe, Ph.D. Suggested by Quest. 4269, Vol. liii., p. 131.) — A circle being divided into an *odd* number n, $= 2m + 1$, of equal parts, let $A_1, A_2, A_3, \ldots, A_n$ be the successive points of division, O an arbitrary point of the circumference situated between A_1 and A_n. Join O to $A_1, A_2, A_3, \ldots, A_n$ by the chords $a_1, a_2, a_3, \ldots, a_n$; then prove that

(1) $(a_1 + a_n)(a_2 + a_{n-1})(a_3 + a_{n-2}) \ldots (a_{\frac{1}{2}(n-1)} + a_{\frac{1}{2}(n+3)}) = a_{\frac{1}{2}(n+1)}^{\frac{1}{2}(n-1)}$;

(2) $a_1 + a_n + a_3 + a_{n-2} + a_5 + a_{n-4} + \ldots - (a_2 + a_{n-1} + a_4 + a_{n-3} + a_6 + a_{n-5} + \ldots)$
$= (-1)^{m-1} a_{\frac{1}{2}(n+1)}$;

and find (3) how these relations should be modified if n is an *even* number.

Solutions by R. F. Davis, M.A., *and the* Proposer.

Let $A_1A_2 \ldots A_9$ be a regular nonagon inscribed in a circle; O a point on the circle between A_9A_1 whose distances from the vertices are $a_1, a_2, \ldots a_9$, respectively.

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Let $A_1A_2 = A_2A_3 = \dots = c$, $A_1A_3 = A_2A_4 = \dots = d$.

Then, by repeated applications of Euclid VI. D (Ptolemy's Theorem),

$$\begin{array}{lll} c(a_1+a_3) = da_2 & c(a_4+a_6) = da_5 & c(a_7+a_9) = da_8 \\ c(a_2+a_4) = da_3 & c(a_5+a_7) = da_6 & c(a_8-a_1) = da_9 \\ c(a_3+a_5) = da_4 & c(a_6+a_8) = da_7 & c(a_2-a_9) = da_1 \end{array}$$

Hence $d\{(a_1+a_3+a_5+a_7+a_9)-(a_2+a_4+a_6+a_8)\}$

$$= c\{(a_3-a_9)+(a_2+a_4)+(a_4+a_6)+(a_6+a_8)+(a_8-a_1) - (a_1+a_3)-(a_3+a_5)-(a_5+a_7)-(a_7+a_9)\},$$

or $$(d+2c)\{(a_1+a_3+a_5+a_7+a_9)-(a_2+a_4+a_6+a_8)\} = 0.$$

But $d+2c$ is not $= 0$; hence

$$a_1+a_3+a_5+a_7+a_9 = a_2+a_4+a_6+a_8,$$

or sum of distances of odd vertices is equal to the sum of distances of even vertices. Again,

$$\begin{array}{ll} A_1A_5 \cdot (a_1+a_9) = A_1A_9 \cdot a_5 & A_3A_5 \cdot (a_3+a_7) = A_3A_7 \cdot a_5 \\ A_2A_5 \cdot (a_2+a_8) = A_2A_8 \cdot a_5 & A_4A_5 \cdot (a_4+a_6) = A_4A_6 \cdot a_5 \end{array}$$

Multiplying together, we have

$$(a_1+a_9)(a_2+a_8)(a_3+a_7)(a_4+a_6) = a_5^4;$$

for $A_1A_5 = A_3A_7$, $A_2A_5 = A_2A_8$, $A_3A_5 = A_4A_6$, $A_4A_5 = A_1A_9$.

The above geometrical method does not apparently suggest the analogous theorem for a regular polygon of an even number of sides.

Prof. LAMPE adds the following further developments:—

Let the diameter d of the circumference be unity, and 2α the angle subtended by a_1 at the centre. Then $a_2, a_3, \dots, a_n$ will subtend the angles

$$2\alpha + 2\pi/n, \quad 2\alpha + 4\pi/n, \quad 2\alpha + 6\pi/n, \quad \dots, \quad 2\alpha + (n-1)2\pi/n,$$

respectively; and therefore

$$a_1 = \sin\alpha, \; a_2 = \sin(\alpha+\pi/n), \; a_3 = \sin(\alpha+2\pi/n), \dots, a_n = \sin\{\alpha+(n-1)\pi/n\} \quad \dots\dots\dots\dots\dots\dots (3);$$

whence
$$\left.\begin{array}{rcl} a_1 + a_n &=& 2\sin\{\alpha+(n-1)\pi/2n\} \cdot \cos(n-1)\pi/2n \\ a_2 + a_{n-1} &=& 2\sin\{\alpha+(n-1)\pi/2n\} \cdot \cos(n-3)\pi/2n \\ a_3 + a_{n-2} &=& 2\sin\{\alpha+(n-1)\pi/2n\} \cdot \cos(n-5)\pi/2n \\ \vdots \quad\quad \vdots & & \vdots \quad\quad\quad\quad \vdots \\ a_{\frac{1}{2}(n-1)} + a_{\frac{1}{2}(n+3)} &=& 2\sin\{\alpha+(n-1)\pi/2n\} \cdot \cos 2\pi/2n \end{array}\right\} \dots (4),$$

$$a_{\frac{1}{2}(n+1)} = \sin\{\alpha+(n-1)\pi/2n\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5).$$

I. Putting $n = 2m+1$ in the second factors of the right-hand sides of (4), and multiplying, we get

$$(a_1+a_n)(a_2+a_{n-1})(a_3+a_{n-2}) \dots (a_{\frac{1}{2}(n-1)} + a_{\frac{1}{2}(n+3)})$$

$$= a_{\frac{1}{2}(n+1)}^{\frac{1}{2}(n-1)} \cdot 2^m \cos\pi/n \cdot \cos 2\pi/n \dots \cos m\pi/n \quad \dots\dots (6).$$

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But, after a well-known formula,

$$2^m \cdot \cos\frac{\pi}{2m+1} \cdot \cos\frac{2\pi}{2m+1} \cdot \cos\frac{3\pi}{2m+1} \ldots \cos\frac{m\pi}{2m+1} = 1,$$

wherefore (6) changes into

$$(a_1 + a_n)(a_2 + a_{n-1}) \ldots (a_{\frac{1}{2}(n-1)} + a_{\frac{1}{2}(n+3)}) = a_{\frac{1}{2}(n+1)}^{\frac{1}{2}(n-1)} \quad \ldots\ldots\ldots\ldots(1).$$

Corollary 1.—The relation (1) admits of n different forms, if any of the chords a_i is taken as the first; but then the signs of the chords in the binomials must be chosen conveniently, after a rule which may be easily pointed out. We need only change the indices 1, 2, 3, ..., n by cyclic permutation, and put the sign $-$ as soon as the highest index n will be exceeded. Take, for instance, $n = 5$:—

$$\begin{aligned}
&(a_1 + a_5)\quad (a_2 + a_4) = a_3^2,\\
&(a_2 - a_1)\quad (a_3 + a_5) = a_4^2,\\
&(a_3 - a_2)\quad (a_4 - a_1) = a_5^2,\\
&(a_4 - a_3)\quad (a_5 - a_2) = a_1^2 = (a_3 - a_4)(a_2 - a_5),\\
&(a_5 - a_4)(-a_1 - a_3) = a_2^2 = (a_4 - a_5)(a_1 + a_3).
\end{aligned}$$

For $n = 7$, there will result the following table :—

$$\begin{aligned}
&(a_1 + a_7)\quad (a_2 + a_6)\quad (a_3 + a_5) = a_4^3,\\
&(a_2 - a_1)\quad (a_3 + a_7)\quad (a_4 + a_6) = a_5^3,\\
&(a_3 - a_2)\quad (a_4 - a_1)\quad (a_5 + a_7) = a_6^3,\\
&(a_4 - a_3)\quad (a_5 - a_2)\quad (a_6 - a_1) = a_7^3,\\
&(a_5 - a_4)\quad (a_6 - a_3)\quad (a_7 - a_2) = -a_1^3,\\
&(a_6 - a_5)\quad (a_7 - a_4)(-a_1 - a_3) = -a_2^3,\\
&(a_7 - a_6)(-a_1 - a_5)(-a_2 - a_4) = -a_3^3.
\end{aligned}$$

II. Summing up the equations (4) taken with alternate signs, we obtain

$$\begin{aligned}
S = {} & (a_1 + a_n) - (a_2 + a_{n-1}) + (a_3 + a_{n-2}) - (a_4 + a_{n-5}) + \ldots \\
& \ldots + (-1)^{m-1}(a_{\frac{1}{2}(n-1)} + a_{\frac{1}{2}(n+3)}) \\
= {} & 2a_{\frac{1}{2}(n+1)} \left\{ \cos\frac{m\pi}{2m+1} - \cos(m-1)\frac{\pi}{2m+1} + \cos(m+2)\frac{\pi}{2m+1} - \ldots \right. \\
& \left. \ldots + (-1)^{m-1}\cos\frac{\pi}{2m+1} \right\}.
\end{aligned}$$

(a) m is an odd number. Change the negative terms

$$-\cos(m-1)\frac{\pi}{2m+1} \text{ into } \cos\left\{\pi - (m-1)\frac{\pi}{2m+1}\right\} = \cos(m+2)\frac{\pi}{2m+1}, \text{ \&c.,}$$

and the sum between the brackets proves to be

$$\Sigma = \cos\frac{\pi}{2m+1} + \cos\frac{3\pi}{2m+1} + \cos\frac{5\pi}{2m+1} + \ldots + \cos(2m-1)\frac{\pi}{2m+1}.$$

From the formula

$$\cos\beta + \cos 3\beta + \cos 5\beta + \ldots + \cos(2m-1)\beta = \frac{\sin 2m\beta}{2\sin\beta}$$

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results immediately

$$\Sigma = \left(\sin\frac{2m\pi}{2m+1}\right) \Big/ \left(2\sin\frac{\pi}{2m+1}\right) = \tfrac{1}{2},$$

because $$\sin\frac{2m\pi}{2m+1} = \sin\frac{\pi}{2m+1}.$$

Substituting this value, we have

$$S = a_{\frac{1}{2}(n+1)} = (-1)^{m-1} a_{\frac{1}{2}(n+1)}.$$

(β) m is an even number. In this case we may write for the positive terms $\cos\frac{m\pi}{2m+1} = -\cos\left(\pi - \frac{m\pi}{2m+1}\right) = -\cos(m+1)\frac{\pi}{2m+1}$, &c.,

and the sum between the brackets is transformed into

$$-\Sigma, \quad = -\tfrac{1}{2},$$

whence $$S = -a_{\frac{1}{2}(n+1)} = (-1)^{m-}\ a_{\frac{1}{2}(n+1)}.$$

Thus the relation (2) is proved to be true.

Corollary 2.—For $n = 3$ the relation (2) furnishes the well-known proposition: The sum of the chords joining a point O of a circumference to the two adjacent vertices of an inscribed equilateral triangle, is equal to the chord joining O with the third vertex.

Particular theorems flow from (1) and (2) if O is chosen midway between A_1 and A_n on the circumference.

Complementary Question, when n = an even number.

I. In this case the equations (4) may be written

$$\left.\begin{aligned} a_1 + a_n &= 2\sin\{\alpha + (n-1)\pi/2n\}\cdot\cos(n-1)\pi/2n \\ a_2 + a_{n-1} &= 2\sin\{\alpha + (n-1)\pi/2n\}\cdot\cos(n-3)\pi/2n \\ \vdots \quad \vdots &\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{\frac{1}{2}n} + a_{\frac{1}{2}n+1} &= 2\sin\{\alpha + (n-1)\pi/2n\}\cdot\cos\pi/2n \end{aligned}\right\} \ldots\ldots(4a).$$

Consequently,

$$(a_1+a_n)(a_2+a_{n-1}) \ldots (a_{\frac{1}{2}n} + a_{\frac{1}{2}n+1})$$
$$= \sin^{\frac{1}{2}n}\{\alpha + (n-1)\pi/2n\}\cdot 2^{\frac{1}{2}n}\cos\pi/2n.\cos 3\pi/2n \ldots\cos(n-1)\pi/2n \ldots (6a).$$

But, from the formula for $\cos nx$, it is evident that

$$2^{\frac{1}{2}n}\cdot\cos\pi/2n\cdot\cos 3\pi/2n \ldots \cos(n-1)\pi/2n = 2^{\frac{1}{2}};$$

moreover $$\sin\{\alpha + (n-1)\pi/2n\} = \cos(\alpha - \pi/2n),$$

and we deduce, from (6a),

$$(a_1+a_n)(a_2+a_{n-1}) \ldots (a_{\frac{1}{2}n} + a_{\frac{1}{2}n+1}) = \cos^{\frac{1}{2}n}(\alpha - \pi/2n)\, 2^{\frac{1}{2}} \quad \ldots\ldots(1a),$$

where $\cos(\alpha - \pi/2n)$ admits of an easy construction. It is the chord O′P joining the extremity O′ of the diameter OO′ passing through O, to the point P bisecting the arc A_1A_n. The equation (1a) is the relation required, analogous to (1).

II. $$S = (a_1+a_n) - (a_2+a_{n-1}) + (a_3+a_{n-2}) - \ldots + (-1)^{\frac{1}{2}n-1}(a_{\frac{1}{2}n} + a_{\frac{1}{2}n+1})$$
$$= 2\sin\{\alpha + (n-1)\pi/2n\}\{\cos(n-1)\pi/2n - \cos(n-3)\pi/2n + \ldots$$
$$\ldots + (-1)^{\frac{1}{2}n-1}\cos\pi/2n\}.$$

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Now we have the formula

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+m\beta)$$
$$=\frac{\cos(\alpha+\frac{1}{2}m\beta)\sin\frac{1}{2}(m+1)\beta}{\sin\frac{1}{2}\beta}.$$

Consequently,

$$\cos\pi/2n+\cos 5\pi/2n+\cos 9\pi/2n+\ldots+\cos(2n-3)\pi/2n$$
$$=\frac{\cos\{\pi/2n+(\frac{1}{2}n-1)\pi/n\}\sin\frac{1}{2}n\,.\,\pi/n}{\sin\pi/n}$$
$$=\frac{-\sin\pi/2n}{2\sin\pi/2n\,.\,\cos\pi/2n}=-\frac{1}{2\cos\pi/2n},$$

and $(a_1+a_n)-(a_2+a_{n-1})+(a_3+a_{n-2})-(a_4+a_{n-3})+\ldots$

$$=\frac{(-1)^{\frac{1}{2}n-1}}{\cos\pi/2n}=(-1)^{\frac{1}{2}n-1}\sec\pi/2n\ldots\ldots\ldots\ldots(2a),$$

the analogue of (2).

[We may remark that there are many other relations resulting from the formulæ for the trigonometric series or the equations for the values of $\cos mx$, $\sin nx$. Take, for instance,

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}$$
$$=\frac{\sin\{\alpha+(n-1)\frac{1}{2}\beta\}\sin n\frac{1}{2}\beta}{\sin\frac{1}{2}\beta},$$

then it will become $a_1+a_2+a_3+\ldots+a_n=\dfrac{\cos(\alpha-\pi/2n)}{\sin\pi/2n}$ .....(7).

If we start from Euler's formula (*Introductio in Analysin*, Cap. XIV., Lib. I., § 240),

$\sin nz=2^{n-1}\sin z\sin(\pi/n-z)\sin(\pi/n+z)\sin(2\pi/n-z)\sin(2\pi/n+z)\ldots$;

we shall obtain $\sin n\alpha=2^{n-1}a_1,\ a_2,\ a_3\ldots a_n$ .....(8).

If we wish to have homogeneous relations, we may add convenient powers of the diameter d, say d^n in the left-hand side of (8), and the like.]

11599 and **11670.** (R. Tucker, M.A.)—11599. In a triangle ABC, points are taken on the sides such that

$$BP:CP=CQ:AQ=AR:BR=m:n=CP':BP'$$
$$=AQ':CQ'=BR':AR';$$

prove that (1) radical axis of circles PQR, P′Q′R′ passes through the centroid and "S." points of ABC; (2) if QR, Q′R′ cut in 1, RP, R′P, in 2, PQ, P′Q′ in 3, then the equation to the circle 123 is

$$abc\,\Sigma a\beta\gamma=mn\Sigma a\alpha\,.\,\Sigma a\alpha\left\{-mna^2+(m^2+mn+n^2)(b^2+c^2)\right\};$$

also (3) indicate other properties of the system.

11670. In Question 11599, (3) prove the conic about P′PQ′QR′R to be concentric, similar, and similarly situated with the minimum circum-

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ellipse of ABC; (4) the "S." points of PQR, P'Q'R' are

$$\frac{aa}{mq^2+nr^2} = \ldots = \ldots; \quad \frac{aa}{nq'^2+mr'^2} = \ldots = \ldots;$$

where p = QR, ...; p' = Q'R', ...; and the "S." point of 123 is

$$aa/\left[(m^2+n^2)\,a^2+mn\,(b^2+c^2)\right] = \ldots = \ldots .$$

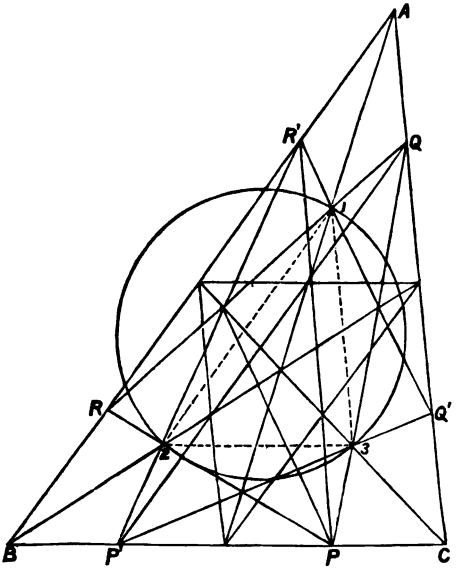
Solution by the PROPOSER.

The points P, Q, R are given by

$$(0, nc, mb), \quad (mc, 0, na), \quad (nb, ma, 0),$$

i.e., P, in trilinear coordinates, is (0, $nc \sin A$, $mb \sin A$), &c.; and P', Q', R' by

$$(0, mc, nb), \quad (nc, 0, ma), \quad (mb, na, 0).$$



It is, hence, evident that the pairs of triangles are concentroidal with each other and with ABC.

It is also evident that PQ', P'Q are parallel to AB, and so on; also that P'Q, PR' intersect on the median through A, and so on.

The triangle PQR = $(m^2-mn+n^2)\,\Delta$ = the triangle P'Q'R'.

The equation to the circle PQR is

$$(m^2-mn+n^2)\,abc\,.\,\Sigma(a\beta\gamma) = mn\Sigma(aa).\Sigma(aa\,.-mna^2+m^2b^2+n^2c^2),$$

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and to P′Q′R′ is

$$(m^2-mn+n^2)\,abc\,.\,\Sigma(\alpha\beta\gamma) = mn\Sigma(a\alpha).\Sigma(a\alpha\,.\,-mna^2+n^2b^2+m^2c^2).$$

The radical axis of these circles is, therefore,

$$\Sigma(a\alpha\,.\,b^2-c^2) = 0, \text{ hence } \ldots\ldots\ldots\ldots (1).$$

The radical axis of either of the circles and of the circumcircle is of the form $l^2P-lQ+R=0$, where P, Q, R are linear functions of α, β, γ; and the envelope of each of these axes is the conic

$$(a^3\alpha+b^3\beta+c^3\gamma)^2 = 4\,(ab^2\alpha+bc^2\beta+ca^2\gamma)\,(ac^2\alpha+ba^2\beta+cb^2\gamma) \ldots\ldots(a).$$

The tangents in (a) intersect in the point $a\alpha/(a^4-b^2c^2) = \ldots = \ldots$.

The radical centre of the three circles is

$$a\alpha/\left[a^4-b^2c^2+mn\mathrm{K}\,(\mathrm{K}-3a^2)\right] = \ldots = \ldots;$$

where $$\mathrm{K} \equiv a^2+b^2+c^2.$$

The equations to QR, Q′R′ are

$$\left.\begin{aligned} -mna\alpha+n^2b\beta+m^2c\gamma = 0 \\ -mna\alpha+m^2b\beta+n^2c\gamma = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (b);$$

and 1, their point of intersection, is on the median through A, and is given by $$a\alpha/(m^2+n^2) = b\beta/(mn) = c\gamma/mn.$$

Similarly 2, 3 are $a\alpha/mn = b\beta/(m^2+n^2) = c\gamma/mn$,

$$a\alpha/mn = b\beta/mn = c\gamma/(m^2+n^2).$$

The above lines (b) envelope the parabola $a^2a^2 = 4bc\beta\gamma$, and so on.

The triangle 123 is readily found to be

$$= (m^2-mn+n^2)^2\Delta.$$

The circle 123 has its equation

$$abc\,.\,\Sigma(a\beta\gamma) = mn\Sigma(a\alpha).\Sigma\left\{a\alpha.\,-mna^2+(m^2+mn+n^2)\,(b^2+c^2)\right\} \ldots(2).$$

The radical axis of this circle and the circumcircle can be written

$$(1-mn)\,\mathrm{K}\Sigma(a\alpha) = \Sigma(a^3\alpha),$$

hence it is a straight line parallel to the chord of contact of the conic (a).

The lines PR, P′Q′, ... intersect in 4, 5, 6, given by

$$a\alpha/(mn-n^2) = b\beta/m^2 = c\gamma/m^2, \ldots,$$

showing that these points are also on the medians, as is evident from the symmetry of the figure.

The lines PR′, P′Q, ... intersect in p, q, r, where p is given by

$$a\alpha/(m-n) = b\beta/n = c\gamma/n.$$

The conic through PP′QQ′RR′ has for its equation

$$mn\,(a\alpha+b\beta+c\gamma)^2 = bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta \ldots\ldots\ldots\ldots(4),$$

which, in the figure, is an ellipse, concentric, similar and similarly situated with the minimum circum-ellipse of ABC.

The polar of A, with regard to (4), is

$$2amna-(m^2+n^2)\,(b\beta+c\gamma) = 0,$$

therefore it is parallel to BC, and cuts AC in J (say); so that $AJ = 2mn\,.\,AC$. The triangle formed by the three polars (for A, B, C) is

$$= 4\,(m^2-mn+n^2)^2\Delta.$$

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The tangents to the conic at P, P′ are given by

$$a\alpha(m^2+n^2)+b\beta m(m-n)-c\gamma n(m-n)=0,$$
$$a\alpha(m^2+n^2)-b\beta n(m-n)+c\gamma m(m-n)=0,$$

and intersect, on the median through A, in the point

$$\frac{a\alpha}{-(m-n)^2}=\frac{b\beta}{m^2+n^2}=\frac{c\gamma}{m^2+n^2};$$

and the triangle formed by this and the corresponding points equals the above triangle.

To find the "S." points of PQR, P′Q′R′, assume the sides of these triangles to be p, q, r; p', q', r'; then

$$\left.\begin{aligned} p^2 &= m^2c^2+n^2b^2-2mnbc\cos A \\ p'^2 &= m^2b^2+n^2c^2-2mnbc\cos A \end{aligned}\right\} \text{ \&c. = \&c.};$$

$$\therefore\quad \Sigma(p^2)=(m^2-mn+n^2)(a^2+b^2+c^2)=K\text{ (suppose)}=\Sigma(p'^2).$$

The "S." lines through Q, R, respectively, are

$$\begin{vmatrix} \alpha & \beta & \gamma \\ nbcr^2 & ca(mr^2+np^2) & mabp^2 \\ mc & o & na \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta & \gamma \\ mbcq^2 & ncap^2 & (mp^2+nq^2)ab \\ nb & ma & o \end{vmatrix},$$

i.e.,
$$-naa(mr^2+np^2)+(n^2r^2-m^2p^2)b\beta+mc\gamma(mr^2+np^2)=0,$$
$$maa(mp^2+nq^2)-(mp^2+nq^2)nb\beta-c\gamma(m^2q^2-n^2p^2)=0;$$

whence we get, for the "S." point of PQR (K_1),

$$\frac{a\alpha}{mq^2+nr^2}=\frac{b\beta}{mr^2+np^2}=\frac{c\gamma}{mp^2+nq^2}=\frac{2\Delta}{K}.$$

Similarly, for the "S." point of P′Q′R′ (K_2), we have

$$\frac{a\alpha}{nq'^2+mr'^2}=\frac{b\beta}{nr'^2+mp'^2}=\frac{c\gamma}{np'^2+mq'^2}=\frac{2\Delta}{K}.$$

The triangle 123 is directly in perspective with ABC, and has the centroid of the triangles for centre of perspective; hence we can readily obtain the coordinates of the principal points.

For (1) the "S." point

$$a\alpha/\left[a^2+mn(b^2+c^2+2a^2)\right]=\ldots=\ldots;$$

(2a) the positive "B." point

$$a\alpha/\left[(m^2+n^2)c^2a^2+mnb^2(c^2+a^2)\right]=\ldots=\ldots;$$

(2b) the negative "B." point

$$a\alpha/\left[(m^2+n^2)a^2b^2+mnc^2(a^2+b^2)\right]=\ldots=\ldots;$$

(3) the in-centre $a\alpha/\left[a(m^2+n^2)+(b+c)mn\right]=\ldots=\ldots;$

(4) the orthocentre

$$\alpha/\left[(m^2+n^2)\cos B\cos C+mn\cos A\right]=\ldots=\ldots;$$

(5) the circumcentre

$$\alpha/\left[(m^2+n^2)\cos A+mn\cos(B-C)\right]=\ldots=\ldots.$$

It is readily seen that the lines (AP, BQ), (AP′, BQ′) intersect on the

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conic $c^2\gamma^2 = ab\alpha\beta$, which touches CA, CB at A and B, and passes through the centroid.

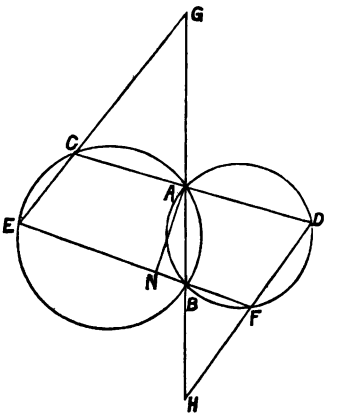
The coordinates of the centre are $\{\frac{1}{3}(2c \sin B), \frac{1}{3}(2c \sin A), \frac{1}{3}(-a \sin B)\}$; like results hold for the other points of intersection.

11606. (A. Kahn, B.A.)—Two circles intersect in A, B. Through A any line CAD is drawn intersecting the circles in C and D; and through B the line EBF is drawn parallel to CD, and cutting the circles in E and F. EC and BA are produced to meet in G, and FD and BA are produced to meet in H. Prove that the square of the tangent from G to either circle is to the square of the tangent from H as $EA^2 - AB^2$ is to $FA^2 - AB^2$.

Solution by J. Rice; Professor Zerr; *and others.*

Draw AN perpendicular to EF;
then $\angle ADF$ = supplement ABF
= supplement ACE;
therefore CDFE is a parallelogram,
and $\angle GBE$ = supplement ACE
= GEB;
$\therefore$ CA = EN − NB.
Similarly, AD = FN + NB. Now

$$\frac{\text{square on tangent from G}}{\text{square on tangent from H}} = \frac{GA \cdot GB}{HA \cdot HB} = \frac{CA \cdot EB}{AD \cdot FB}$$
$$= \frac{(EN - NB)(EN + NB)}{(FN + NB)(FN - NB)}$$
$$= \frac{EA^2 - AB^2}{FA^2 - AB^2}.$$



11574. (I. Arnold.)—A uniform circular plate rests with its centre O on a fixed point. Three weights P, Q, R are attached to the circumference so as to keep it in equilibrium in a horizontal position; it is required (1) to determine the angles POQ, QOR, ROP in terms of the weights, and without assuming any given diameter; (2) assuming the diameter 20 and the weights P, Q, R, 20, 30, 40, find the sides of the triangle made by the weights, or their points of suspension.

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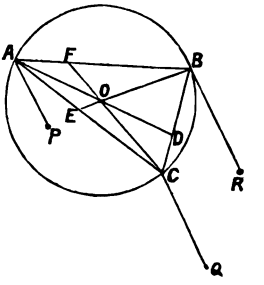
Solution by Professors ZERR, BHATTACHARYA, *and others.*

Angles DOC, EOC, AOE are the angles of a triangle whose sides are proportional to the weights. DOC, EOC, AOE are opposite to R, P, Q; hence



$$\cos \mathrm{DOC} = \frac{P^2+Q^2-R^2}{2PQ};$$

$$\cos \mathrm{AOC} = \frac{R^2-P^2-Q^2}{2PQ};$$

$$\cos \mathrm{AOB} = \frac{Q^2-P^2-R^2}{2PR};$$

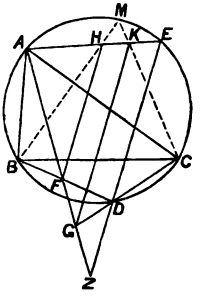
$$\cos \mathrm{BOC} = \frac{P^2-Q^2-R^2}{2QR}.$$

When radius = 10, P = 20, Q = 30, R = 40;

AC = 12·2474487, AB = 18·3711737, BC = 19·3649167.

11697. (Professor BERNES.)—DE étant une corde de la circonférence circonscrite à un triangle ABC; F, G les points de rencontre de DB, DC avec une sécante variable AZ; H, K les points où les parallèles à DE par F, G rencontrent AE; quel est le lieu de l'intersection M des droites BH, CK, lorsque AZ tourne autour de A?

Solution by J. C. ST. CLAIR · F. J. DAVIS, M.A; *and others.*



Let AZ meet DE in Z.

Then the anharmonic ratio

$$(\mathrm{M}.\mathrm{ABCE}) = (\mathrm{AHKE}) = (\mathrm{AFGZ}) = (\mathrm{D}.\mathrm{ABCE});$$

therefore M lies on the circle.

11671. (H. BROCARD.) — Soient AA′, BB′ deux diamètres rectangulaires d'un cercle O. Une sécante quelconque menée par A rencontre BB′ en C; la perpendiculaire élevée en C sur AC coupe AA′ en D; trouver le lieu géométrique du point M, intersection de AC avec BD. En déduire la construction d'un angle dont la tangente soit le carré de la tangente d'un angle donné.

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Solution by J. Dale, M.A.; H. W. Curjel, B.A.; *and others.*

Taking A as the origin of rectangular coordinates, put

$$\angle OAC = \theta;$$

then $CO = a \tan \theta$,

$DO = CO \tan \theta = a \tan^2 \theta$;

coordinates of B, D are

$$(x = a,\ y = a),$$

$$(x = a \sec^2 \theta,\ y = 0);$$

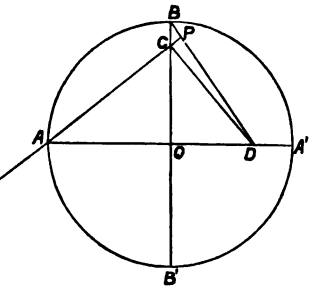
equations to BD, AC are

$$(y-a) \tan^2 \theta = x - a,\ y = x \tan \theta;$$

hence, at P,

$$(y-a)\, y^2 = -\, x^2 (x-a)$$

or $x^3 + y^3 - a(x^2 + y^2) = 0$



is the equation to the locus of P which represents a cubic passing through A, B, and O.

From the figure it is seen that $\tan OBD = \frac{OD}{OB} = \tan^2 \theta$; hence a ready means of constructing an angle of which the tangent shall be equal to the square of the tangent of a given angle.

11676. (J. H. Grace.)—If Q and Q_1 are the centres of curvature of the extremities of a focal chord of a parabola whose focus is S, and if a line parallel to that focal chord meet SQ, SQ_1 in C and D, and the axis of the parabola in B, then SB = CD.

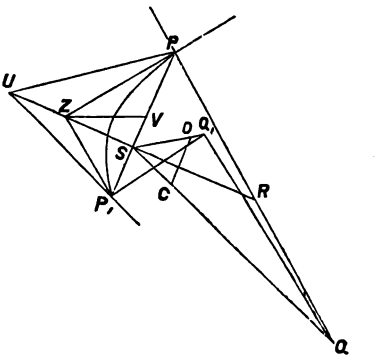
Solution by R. F. Davis, M.A.; R. Knowles, B.A.; *and others.*

Let PSP_1 be a focal chord of a parabola the tangents ZP, ZP_1 at whose extremities intersect at right angles in Z on the directrix, SZ being perpendicular to PSP_1, and the diameter through Z bisecting PP_1 in V. If the normal at P (parallel to ZP_1) intersect ZS in R, and be produced to Q, so that PR = RQ, then Q is the centre of curvature at P.

Produce SZ to U, so that

$$SZ = ZU;\ \text{then}$$

$$QR : SR = PR : SR = ZP_1 : SZ = ZP_1 : UZ;$$



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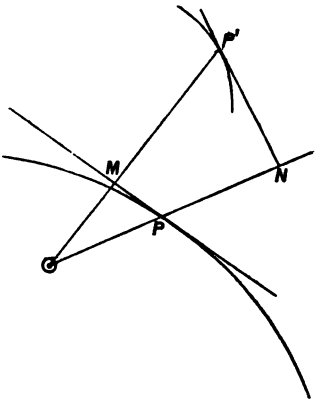
therefore UP′ is parallel to SQ. Similarly, UP is parallel to SQ_1; and the triangles UPP_1, SDC are similar. But a line through U parallel to the axis to meet PP′ = 2ZV = PP′. Whence, &c.

11612. (A. E. Jolliffe.)—A particle describes freely a curve under the action of a force towards a point O. Show that the reciprocal of this curve with respect to O can be described freely by a particle under the action of a force, towards O, proportional to the reciprocal of the normal force in the first orbit.

Solution by J. Burke, M.A.; Professor Mukhopadhyay; *and others.*

Let P and P′ be two corresponding points on the two curves, respectively, PM and P′N being the tangents at P and P′, F and F′ the forces at P and P′ towards O.



Let $OP = r$, $OP' = r'$,

$OM = p$, $ON = p'$.

Then $$F' \propto \frac{1}{p'^3} \cdot \frac{dp'}{dr'};$$

but $$pr' = p'r = k^2,$$

$$pdr' + r'dp = 0, \quad p'dr + rdp' = 0;$$

therefore

$$F' \propto p^2 r \frac{dr}{dp} \propto \frac{1}{\frac{p}{r}\frac{dp}{p^3 dr}} \propto \frac{1}{F \sin OPM}.$$

[An alternative solution is as follows:—If accented letters refer to the reciprocal and unaccented to the original curve, then we have

$$F' \propto u'^2 \left(\frac{d^2u'}{d\theta'^2} + u'\right); \quad \propto p^2 \left(\frac{d^2p}{dw^2} + p\right) \propto \frac{l}{v^2} \propto \frac{1}{N},$$

where N is the normal force.]

11536. (R. Knowles, B.A.)—A circle touches a given parabola S at a point P at an end of the latus-rectum, and cuts it in QR; PC is the chord of curvature at P; K the pole of QR with respect to S; QR meets the tangent at P in M. Prove that (1) the envelope of MK is a parabola S′, which touches S at P and cuts it in Q′R′; (2) C is the pole of Q′R′

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with respect to S′; (3) if K′ is the pole of Q′R′ with respect to S, then Q′R′, the axis of S′, and the perpendicular from K′ on the axis of S are concurrent.

Solution by Rev. T. Galliers, M.A.; Prof. Chakrivarti; *and others.*

1. Let the parabola S be $y^2-4ax=0$; then P is $(a, 2a)$, and the equations of PM and QR respectively are $y-x=a$ and $y+x=c$, where c is a constant; also the equation of MK may be written

$$5ax+ay+3a^2-(3y+a-x)c+c^2=0.$$

Hence the envelope of MK will be

$$x^2-6xy+9y^2-22ax+2ay-7a^2=0 \quad (S'),$$

which represents a parabola; and, since it may be written

$$4(y^2-4ax)=(y-x-a)(x-5y-7a),$$ S′ touches S at P.

2. Again, we see from the above equation that the equation of Q′R′ is

$$x-5y=7a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and, since C is $(9a, -6a)$, C is the pole of Q′R′ with respect to S′.

3. The equations of the axis of S′ and of the perpendicular from K′ on A′ are $5x-15y=7a \quad x=-7a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2, 3)$;

and (1), (2), (3) are satisfied when $x=-7a$, $y=-1\frac{4}{5}a$, which proves the third part of the question.

11393. (H. W. Curjel, B.A.)—If p is a prime not less than 5, show that the number of primes greater than p and less than p^2 is not less than p.

Solution by the Proposer.

If we take the first p^2 integers 1, 2, 3, ... p^2, and take away all that are divisible by 2, 3, 5, ... p, we shall have remaining 1 and the primes greater than p and less than p^2, since any composite number less than p^2 must have a factor less than p. The number of numbers remaining will be

$$\mathrm{I}\left(\frac{p-1}{p}\ldots\mathrm{I}\left\{\frac{5-1}{5}\left[\mathrm{I}\left\{\frac{3-1}{3}\,\mathrm{I}\left(\frac{2-1}{2}p\right)\right\}\right]\right\}\right),$$

where $\mathrm{I}(N/m)$ = the integral part of N/m.

Since the first of the numbers taken away from the series must in each case be in the series 1, 2, 3, 4, 5 ... p, the number of primes remaining is

$$> \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\ldots\frac{p-1}{p}(p^2-p),$$

much more then $> \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\ldots\frac{p-1}{p}(p^2-p)$;

i.e., $> p-1$; and therefore $\not< p$; therefore number of primes greater than p and less than p^2 is not less than p.

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11417. (Professor GELIN.)—Dans tout triangle ABC on a

$$(a\tan B - 2\gamma)(b\tan C - 2\alpha)(c\tan A - 2\beta) = -(a\tan C - 2\beta)(b\tan A - 2\gamma)(c\tan B - 2\alpha),$$

α, β, γ étant les rayons des cercles ex-inscrits respectivement dans les angles A, B, C.

Solution by W. J. GREENSTREET, M.A.; R. KNOWLES, B.A.; *and others.*

$$a\tan B - 2\gamma = a\tan B - \frac{2\Delta}{s-c} = a\tan B - \frac{ac\sin C}{s-c} = \frac{a\tan B}{s-c}(s - 2c\cos^2\tfrac{1}{2}B);$$

therefore, &c.

11353. (H. W. CURJEL, B.A.)—A soap bubble B, of radius b, is inside another bubble A, of radius a. The bubbles are acted on by no external forces except the pressure of the air. Find the effect of the bursting (1) of A, (2) of B, on the size of the other, the surface tension of the soap solution and the atmospheric pressure being known, and the thickness of the film being neglected.

Solution by the PROPOSER.

Let Π be the atmospheric pressure, and $\frac{1}{2}t$ the surface tension, and p_1, p_2 the pressures inside the bubbles A and B.

$$\frac{p_1-\Pi}{t} = \frac{2}{a}; \quad \therefore p_1 = \frac{2t}{a} + \Pi; \qquad \frac{p_2-p_1}{t} = \frac{2}{b}; \quad \therefore p_2 = \frac{2t}{a} + \frac{2t}{b} + \Pi.$$

Let ρ, the density of the air, $= kp$, where p is the pressure; therefore mass of air in B

$$= \frac{4\pi k}{3} p_2 b^3 = \frac{4\pi k}{3} b^3 \left(\frac{2t}{a} + \frac{2t}{b} + \Pi\right),$$

and mass of air in A and B together

$$= \frac{4\pi k}{3}\left\{p_1(a^3 - b^3) + p_2 b^3\right\} = \frac{4\pi k}{3}(2ta^2 + 2tb^2 + \Pi a^3).$$

(1) Let x = radius of B after A has burst, and p = pressure inside it; therefore $\frac{p-\Pi}{t} = \frac{2}{x}$; and, since mass of air in B is unaltered,

$$\frac{4\pi k}{3} x^3 p = \frac{4\pi k}{3} b^3 \left(\frac{2t}{a} + \frac{2t}{b} + \Pi\right);$$

therefore

$$\Pi x^3 + 2tx^2 = 2tb^2 + 2t\,\frac{b^3}{a} + \Pi b^3.$$

The positive root of this equation is the radius of B after bursting of A.

(2) Let x = radius of A after B has burst, and p = the pressure inside it.

As before $\frac{p-\Pi}{t} = \frac{2}{x}$ and $\frac{4\pi k x^3 p}{3} = \frac{4\pi k}{3}(2ta^2 + 2tb^2 + \pi a^3)$;

therefore radius of A after bursting of B is the positive root of

$$x^3\Pi + 2tx^2 = 2ta^2 + 2tb^2 + \Pi a^3.$$

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APPENDIX.

UNSOLVED QUESTIONS.

103. (Kate Sullivan.) Through a given point within a given angle to draw a straight line such that the intercepted part shall be the least possible.

135. (Kate Sullivan.)—P is a given point in one of the principal axes AB (produced) of a given ellipse; find the position of a secant P*mn*, intersecting the curve at *m*, *n*, such that, by drawing the perpendiculars *mx*, *ny* on AB, the figure *mxyn* shall be maximum.

440. (J. W. Elliott.) — Given two concentric conics; then, if lines be drawn from each point in one of them to touch the other, the chords of contact will touch another curve of the second order.

445. (Septimus Tebay.)—A given sphere is put into a conical glass whose axis is inclined to the horizon at a given angle; required the whole pressure on the surface of the glass.

455. (J. H. Swale.)—In any plane triangle ACB draw BD perpendicular to the opposite side AC, and let T be the point of contact of the inscribed circle with AC; draw TKL parallel to the line bisecting the angle B and meeting BC, BA at K and L; then prove that (1) BK=BL=DT, and TK . TL = DB^2; and (2) the same is also true for each side and its opposite angle.

456. (J. H. Swale.)—Let CT, C*t* be tangents to a circle; from T, one of the points of contact, and O, the centre of the circle, draw any parallels TL, OB to the other tangent at L, B; and to TL apply BK = BL; then prove that BK will be a tangent, and TL . TK = BD^2, BD being a perpendicular on CT.

458. (Septimus Tebay.)—A heavy circular ring slides from a given position on the surface of a smooth sphere, which is also at liberty to slide freely on a smooth horizontal plane. Determine the position of the ring when it is projected from the sphere.

469. (Septimus Tebay.)—Required the probability of the caster winning a single game at hazard, the main being determined by a cast of the dice, and the game played directly out.

48?. (J. H. Swale.)—In the triangle ABC, take I and O, the centres of the inscribed and escribed circle touching the sides CA and CB produced; draw OP perpendicular to CA, also CQ parallel to PI, meeting OP at Q; then 2PQ = the perpendicular CE. Also, if CO cut AB in D, and DP be drawn to meet CQ in K, then QK = QC.

483. (J. H. Swale.)—In the radius CA, produced, of any given circle (C), take any point P and draw the tangent PT; produce CP to Q, making PQ=PT, and draw the tangent QV; drop the perpendicular VR upon CA; then PR = PQ = PT.

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484. (J. H. Swale.)—Let C be the centre, and B any point in the radius AC of the semicircle ARH; draw any line AR to the periphery, and join AR; then $AR^2 : BR^2 - BA^2 :: AC : CB$; or $AR^2 : AB^2 - BR^2 :: AC : CB$; according as B lies *between* or *beyond* A and C.

505. (R. G. Clarke.)—Within a given isosceles inscribe the greatest parabola, the vertex of the latter being at the middle of the base of the triangle.

507. (J. H. Swale.)—In any triangle ABC let BI be drawn, making $\angle ABI = \frac{1}{2}(\angle B \sim \angle A)$; draw CD bisecting the base AB at D, and cutting BI at K; also draw DI perpendicular to BK, and take $IK . KP = ID^2$; then prove that $IB . IP = CD^2$.

508. (J. H. Swale.)—In any triangle ABC, let CK bisect AB at K, and meet BD (making CD = CB) at N; draw CQ, NL perpendicular to BD, and take $LK . LP = LN^2$; then will $AC . CB = BK . BP$.

509. (Thomas Morley.)—Determine the sides of a plane triangle, such that $a \cot A + b \cot B + c \cot C$ = a square number, the sides being in arithmetical progression.

518. (T. T. Wilkinson.)—Two points being given, show that a straight line is also given in position, such that if, from any point whatever in it, two straight lines be drawn to the given points, the difference of their squares shall equal a given space.

526. (Joseph Lawson.)—Of three weights A, B, C, placed at each of the angles of a given triangle, we have given A, B; find the weight C, such that the sum of the lines drawn from the centre of gravity of the whole to each of the weights may be a minimum.

527. (Thomas Morley.)—A train weighs 100 tons, and is going at the rate of 20 miles per hour when the steam is turned off; find how far the train will go, on a level rail, allowing 8 lbs. per ton friction, the effect of the air being neglected.

528. (J. H. Swale.)—If the trajectory described by a comet be a parabola of which the sun is the focus, and any two circles be described through the focus and the vertex, so as to intersect the parabola in two points; the *nearest* distances of their centres from the axis of the parabola will be to each other as the *times* in which the comet describes the segments of the trajectory intercepted between its vertex and the aforesaid points of intersection. Required the demonstration.

529. (J. H. Swale.)—The two straight lines AL, AN are given in position. Let CB be drawn cutting them respectively in C, B so that $AC^2 + AB^2$ equal a given square, and take P the point of bisection of CB. It is required to determine the locus of the point P.

530. (Geometricus.)—In a given circle, inscribe a triangle so that two sides may pass through given points, and that the third side may be a maximum or a minimum.

539. (J. W. Elliott.)—From a fixed point two straight lines are drawn to meet two fixed straight lines in the points A, B, C, D; required the locus of the point of intersection of AC, BD.

540. (J. W. Elliott.)—Two spheres are hung by strings fixed to a given point, and rest against each other. Required the tension of the strings.

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547. (Stephen Watson.)—A uniform circular well, a feet deep, being filled with water, n men are successively employed to empty it: required the depth emptied by each, when an equal share of the work is done by each man.

549. (J. H. Swale.)—If, in the diameter (AB), or in the diameter produced, of a given circle, and on the same side of the centre (C) with A, there be taken a given point (P); and right lines AN, BN, PN be drawn from A, B, P to any point N whatever in the periphery; then will $AN^2 : PN^2 - PA^2 :: BN^2 : BP^2 - PN^2$ be a given constant ratio.

553. (J. H. Swale.)—If, through P, Q, any two points equally distant from the centre (C) of a given circle, there be drawn the diameters AB, A′B′ and from P, Q; A, B; A′, B′ the right lines PN, QN; AN, BN; A′N, B′N to any point N whatever in the periphery, then will $PN^2 - PA^2 : Q'N^2 - QA'^2 :: AN^2 : A'N^2$, and

$$BP^2 - PN^2 : B'Q^2 - QN^2 :: BN^2 : B'N^2.$$

554. (Geometricus.)—If from a point perpendiculars be drawn on the three sides of a given triangle, prove that the area of the triangle formed by joining the feet of the perpendiculars has a constant ratio to the rectangle under the segments of a chord of the circle circumscribed to the given triangle drawn through the point.

564. (J. W. Elliott.)—A cubical box is half filled with water, and placed upon a rough rectangular board; then, if the board be slowly inclined to the horizon, it is required to determine the conditions, so that the box may slide down or topple over.

567. (Μαθηματικος.)—A person sitting in a railway carriage throws a ball to the roof in a direction perpendicular to the floor of the carriage. His hand, at the time of the ball leaving it, is 5 feet from the roof of the carriage. The train moves at the rate of 1 mile in 2 minutes, and the ball at the rate of 10 feet in a second. Find the angle which the course of the ball makes with the plane of the road.

572. (J. W. Elliott.)—Trace the curves represented by the equations

$$x^4 + y^4 = a^2xy \text{ ..... (1)}, \quad x^4 + y^4 = 2axy^2 \text{ ..... (2)}.$$

579. (J. H. Swale.)—The points P, Q in the diameter AB, or in AB produced, of a given circle are given: it is required to draw the right lines PN, QN to N in the periphery, so that the sum or difference of PN and a given line PD, shall have to QN a given ratio.

581. (J. H. Swale.)—The point P is given in the diameter AB of a given circle; it is required to draw a right line through P, meeting the periphery at N, M, and a tangent AH at H, so that HM : HN shall be a given ratio.

582. (J. H. Swale.)—The points P, Q in the diameter AB of a given circle are given; it is required to draw PN, QN to a point N in the periphery, so that the sum or difference of the lines PN, QN shall be equal to a given line PD.

595. (W. Priestley.)—Given the distance between the centres of the inscribed and escribed circles, the difference of the sides and the vertical angle, to construct the triangle.

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596. (W. Grayson.)—A ball, whose elasticity is ϵ, is projected from a given point in the circumference of a circle, and after being reflected from it twice, returns to the given point. Required the direction of projection.

601. (S. Watson.)—ABCDE is a regular pentagon, and its sides EA, CB are produced to meet in F; FD, CE are joined; then, if the triangles CDE, AFB revolve about the common axis DF, the convex surfaces and volumes generated by these triangles will be equal.

606. (J. Oswald.)—A smooth hollow tube is coiled round a cone; investigate the motion of a ball which is allowed to descend in the tube by its own weight, supposing the base of the cone is in contact with a horizontal plane, along which it is moved uniformly whilst the ball descends.

615. (W. Priestley.)—Suppose a person standing in a railway carriage, which moves uniformly at the rate of 25 miles per hour, were to drop a ball to the bottom of the carriage, a height of 5 feet. Find the length of the track described by the ball in its descent.

619. (A. D. Balfour.)—To find a triangle such that the three sides, perpendicular and line bisecting the base from the opposite angle may all be expressed in rational numbers.

628. (S. Watson.)—From a fixed point in the circumference of a given circle draw *any* chord, and upon it, as a diameter, let a circle be described; it is required to find the area of the curve which is the locus of the ultimate intersections of all the circles so drawn.

631. (M. Collins, B.A.)—Required the locus of all stars that have the same precession in right ascension.

635. (Rev. T. Gaskin.)—(1) In the line AB to find two points E, F, such that $EA \times AF = AC^2$ and $BE \times BF = BD^2$, when AC and BD are given; E and F being also on different sides of A and B respectively. (2) To find E, F when E and F are on the same side of A, but on different sides of B. (3) To find E and F when they lie wholly between A and B.

647. (M. Collins, B.A.)—Required the envelope of a plane which moves so that the rectangle under its distances from the centre of a surface of the second order, and from its pole with respect to the same surface, may remain constant.

650. (J. Williams, M.A.)—In the base AB of a triangle find geometrically a point P, such that by joining it with two given points Q, R by right lines intersecting the other two sides AC, CB, at m and n, the figure AmnB shall be a minimum.

654. (Kate Sullivan.)—In the right line joining the centres of two circles given in position, and in the same plane, find *geometrically* a point P such that if tangents Pm, Pn be drawn to the given circles, $Am^2 + Dn^2$ shall be a minimum (A and D being the remote extremities of the diameters that are in the same straight line with respect to P); also, required P when $Am^2 + Dn^2$ is a given magnitude.

659. (Hibernicus.)—By aid of lines not exceeding the second order, inscribe a quadrilateral in a circle given in magnitude and position, so

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that two of its opposite sides shall intersect in a given point, one of the other sides being of a given length, and such that the rectangle contained by straight lines joining two given points with the intersection of the diagonals shall be a given space, or a minimum.

662. (T. T. Wilkinson.)—A circle O, and three right lines A, D, C are given in position; it is required to circumscribe the given circle by a triangle DEF whose angular points lie on the given lines.

674. (S. Younger.)—An ellipse with its major axis vertical is *just* immersed in a fluid; show that the chord of least pressure through the lower focus is determined by $\sin\theta = (1/\epsilon)(1-\epsilon)$, where θ is the inclination of the chord to the axis major, and ϵ the eccentricity of the conic.

678. (J. Dixon, B.C.S.)—Two cones, having their vertices coincident, their axes at right angles, and their surfaces in contact, are cut by a plane parallel to both axes; find the condition that the section shall be a pair of *conjugate hyperbolas*.

685. (S. Younger.)—A plane (A) passes through two points which are given in position, to a plane (B); the intersection of (A) and (B) likewise passes through a given point in (B): determine the inclination of h e planes.

702. (Geometricus.)—Let AFBH and EQG be two circles given in magnitude and position; a point P may be found, and also the position of a right line RS, such that, if any chord APB be drawn through P meeting the circle AFBH in A and B, and the perpendiculars AW and BS be demitted upon the line to be found, and also the tangents AE and BG be drawn to the circle EQG, the continued product of the said perpendiculars and tangents may be constant. (Geometrically.)

706. (J. W. Elliott.)—If a conic be described through four points on a curve of the third order, it will cut the curve again in two other points, and such that the line joining them will pass through a fixed point on the cubic.

711. (Rev. T. Gaskin.)—The game of Solitaire consists of a board which has 37 holes in the following form, and 37 pegs; it is required to obtain the moves for reducing the pegs to 1, in any of the four following cases, viz., when the peg marked 19, 27, 33, or 34 is taken out at first. Any peg can take its neighbour by going over it into a vacant hole either *horizontally* or *vertically*, but not slantways as in the game of drafts.

```
          1  2  3
       4  5  6  7  8
 9 10 11 12 13 14 15
16 17 18 19 20 21 22
23 24 25 26 27 28 29
       30 31 32 33 34
          35 36 37
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714. (J. W. Elliott.)—A prismatic diving-bell of given volume is sunk to a given depth in water; required the volume of air (at its natural density) which must be forced into the bell so that a given portion of its volume may be free from water.

721. (Mortimer Collins.)—From a vessel of wine containing 256 gallons, 64 gallons are drawn off, and the vessel is refilled with water; find how many times this operation must be repeated, so that not more than 1 gallon of pure wine may remain in the vessel.

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726. (Signor Totelli, A.B.)—There are two bags, each containing 4 white and 4 black balls; four are taken at a venture from one of them, and transferred to the other; then, 8 being drawn from the latter, 6 of them prove *white* and 2 *black*: what is the chance that, if another be drawn, it will be *white*? [Note.—The above problem is here proposed in order to obtain the value of the probability, as that given elsewhere has been made the subject of a dispute.]

731. (Amicus.)—Find three numbers such that the sum of every two of them may be a square number, and that the sum of the three squares thus found may be also a square number.

741. (J. W. Elliott.)—Three stars have the same azimuth at each of two observations taken after a given interval, and have their differences of right ascension and distances known; required the latitude.

756. (Matthew Collins, B.A.)—If R, r be the radii of two spheres inscribed in a cone, so that the greater (R) may touch the less and also the base of the cone, find the volume of the cone.

764. (Mortimer Collins.)—Form the equations whose roots are the squares of the differences of the roots of $x^3-7x+7=0$.

769. (J. W. Elliott.)—A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time, if each person receives a peck every week; but, during the distribution, it is found that one person dies every week, and the corn in consequence lasts *twice* as long as was expected: required the quantity of corn, and the time it lasted.

785. (Mortimer Collins.)—The captain of a steamer, carrying despatches, finds himself 24 miles from the nearest point on the shore, which point is distant 50 miles along the coast from the town he has to reach. The speed of the steamer is 12 miles per hour; an express on shore can be obtained which will travel 15 miles an hour: where ought he to land his despatches so as to convey them to their destination in the least possible time?

797. (Mortimer Collins.)—A uniform heavy chain 1000 feet long is hung from two points in the same horizontal plane 985 feet asunder; find the tension at the lowest point of the curve.

800 (Mortimer Collins.)—Solve the functional equations

$$\phi(\psi x) = 16x, \qquad \psi(\phi x) = x^4.$$

801. (S. Younger.)—A straight rod, of uniform density and given length, is placed inside a smooth paraboloid of revolution having its vertex downwards and axis vertical; find the position in which the rod will rest.

807. (S. Younger.)—A smooth sphere rests on a given inclined plane, against a ledge at right angles to the plane; find the diameter of the sphere when it is on the point of slipping over the ledge.

808. (S. Watson.)—With a given point on the circumference of a given circle as a centre, describe an arc meeting it in two points; find the radius of this arc when the moment of inertia of the arc, revolving about the given point, is a maximum.

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810. (Mortimer Collins.)—What is the greatest number of times in a century that February can have 5 Sundays? In what years of what century did this last happen?

822. (Matthew Collins, B.A.)—If Newton's 22nd Species—viz., the *hyperbola redundans cum tribus diametris*—has a conjugate point, show that this point must be the centre of gravity of the triangle formed by the three asymptotes; and show, also, that Newton's 23rd Species can have no conjugate point.

830. (M. Collins, B.A.)—From a point in a line of the third order that has a double point, only two tangents can be drawn to the curve; prove that the lines drawn from the double point to the points of contact, together with the two tangents at the double point, form a harmonic pencil.

833. (G. Sanderson.)—Find the greatest number of globes, of 1 inch in diameter, which can circumscribe another globe 1 foot in diameter.

844. (M. Collins, B.A.)—Find three sets of values of x and y in whole numbers to answer both the equations x^2+y^2 = square, and x^2+Ay^2= square, when A = 11 or when A = 61.

858. (M. Collins, B.A.)—If a line of the third order has one osculating asymptote (*i.e.*, one which *does not cut* the curve), prove *geometrically* that there exists a conic hyperbola along the other two asymptotes osculating the curve so closely at infinity that it *does not cut* it elsewhere; and show also why such an osculating conic does not exist in other cases.

862. (Editor.)—Three points being given, it is required to find a fourth point, in the same plane, such that its distance from each of the given points may have one to one another a given ratio.

863. (Editor.)—If a conical glass, whose altitude is a and the generating angle θ, be filled with water, find the radius of the sphere which, being put into it, shall cause the greatest quantity of water to overflow.

867. (Mortimer Collins.)—A field is divided into two equal parts, to be reaped by A. and B. B. begins an hour earlier than A., and at noon has done the same fraction of his share as both together have done of the whole field at 11 a.m. Between 12 and 1 they do no work; at 2 p.m. B. is called away, C. and D., each of whom works as fast as A., coming to supply his place, and the two halves are finished simultaneously at 4 p.m. At what hour did A. begin?

873. (Mortimer Collins.)—A point being taken within or without a circle, and in every chord passing through it a second point being taken, such that the chord produced is harmonically divided by these two points and the circumference, determine the locus of the latter points.

876. (Mortimer Collins.)—A person, whose veracity we are unable to determine, asserts that an event has taken place whose simple probability is = $\frac{7}{13}$. Show that the probability of his statement being correct = $\frac{52567144}{52567144}$.

879. (Editor.)—A, B are two fixed points, and C, D two variable points in the circumference of a given circle ABDC; show that

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if the lines AC, BD meet in E, and AD, BC in F, the line EF always passes through a fixed point.

963. (Editor.)—A card is folded in two along a diagonal; determine the distance apart of the other two corners, when the two parts of the card contain a given angle.

988. (Mathematicus.)—Show that if C be the circumference of a circle, d its diameter, and c the chord of an arc a, then

$$c = \frac{16ad\,(C-a)}{5C^2-4a\,(C-a)}, \text{ nearly.}$$

989. (M. Collins, B.A.)—If $a = \sqrt{N}$ nearly, prove that $\sqrt{N}$ will be

$$= \frac{(n+1)\,N+(n-1)\,a^n}{(n-1)\,N+(n+1)\,a^n} \times a, \text{ more nearly.}$$

Prove that each new application of this well known formula will give a new root, having, in general, three times as many correct figures as the foregoing approximate root from which it is found.

992. (S. Watson.)—A circle and square have the same perimeter, and the latter is rolled round the circumference of the former; find the area of the locus of its centre.

994. (Editor.)—A cube of density ρ has a concentric spherical cavity filled with fluid of density ρ'. A material particle, if placed in the centre of the mass, will be held in equilibrium with respect to the forces of attraction of the mass; but if slightly displaced from the centre, determine the nature of the motion, neglecting the resistance of the fluid.

995. (Editor.)—A box contains 10 sovereigns, 20 shillings, and 40 farthings; what is the probability that in five trials a person will draw half the sovereigns, each coin being replaced after it is drawn? Find the value of his expectation in six trials, if he receives a guinea for every sovereign that he draws, but pays 5*s.* for every shilling, and 1*s.* 6*d.* for every farthing drawn.

1015. (Hibernicus.)—Required the position of the right line AB, intersecting two straight lines CA, CB given in position at A and B, and touching a fixed circle at P; so that, if CD be perpendicular to AB, and Q a given point, AD shall have to QB a given ratio. And give a purely geometrical analysis.

1019. (J. Dixon.)—A cubic inch of glass is blown into the form of a globe that will hold one pint of wine. Required the thickness of the glass.

1031. (Editor.)—A solid is formed by the revolution of the Cissoid of Diocles about the diameter ($= a$) of the generating circle. Show that the attraction of the solid upon a particle at its vertex is equal to that of a sphere, whose diameter is a, upon a particle at its surface.

1032. (Editor.)—The extremities of two equal beams are connected by a hinge; they are then placed across a rough cylinder. Determine the locus of the hinge or vertex corresponding to all the positions in which the beams are just on the point of slipping.

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